

Sta 711: Homework 11

Martingales

A sequence $\{(X_n, \mathcal{F}_n), n \geq 0\}$ of random variables $X_n \in L_1(\Omega, \mathcal{F}, \mathbf{P})$ and a nested sequence of σ -algebras $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}$ is a *martingale* if for each $0 \leq n \leq m < \infty$

$$X_n = \mathbf{E}[X_m \mid \mathcal{F}_n],$$

so on average the sequence neither increases nor decreases. Note this implies that X_n is \mathcal{F}_n -measurable. By the tower property of conditional expectations, it is enough to check this condition for $m = n + 1$.

1. A sequence $\{(X_n, \mathcal{F}_n), n \geq 0\}$ is *predictable* if X_{n+1} is \mathcal{F}_n -measurable for each n . Show every predictable martingale is constant (*i.e.*, $X_n = X_0$ a.s.).
2. A sequence $\{(X_n, \mathcal{F}_n), n \geq 0\} \subset L_1(\Omega, \mathcal{F}, \mathbf{P})$ is a *submartingale* if $\mathcal{F}_n \subseteq \mathcal{F}_m \subseteq \mathcal{F}$ and $X_n \leq \mathbf{E}[X_m \mid \mathcal{F}_n]$ for each $0 \leq n \leq m < \infty$ —so, on average, X_n is increasing.

Let $\{(X_n, \mathcal{F}_n), n \geq 0\}$ and $\{(Y_n, \mathcal{F}_n), n \geq 0\}$ be submartingales on $(\Omega, \mathcal{F}, \mathbf{P})$. Show that their max $(X_n \vee Y_n)$ and sum $(X_n + Y_n)$ are submartingales too.

3. Fix $0 < p < 1$, set $q := 1 - p$, and let $\{\xi_j\}$ be iid random variables with $\mathbf{P}[\xi_j = 1] = p$ and $\mathbf{P}[\xi_j = -1] = q$. Set

$$S_n := \sum_{j \leq n} \xi_j,$$

a random walk (possibly an asymmetric one) on the integers starting at $S_0 = 0$.

- (a) For which $\alpha \in \mathbb{R}$ is $X_n := [S_n - \alpha n]$ a martingale?
- (b) For which $\alpha, \beta \in \mathbb{R}$ is $Y_n := [(S_n - \alpha n)^2 - \beta n]$ a martingale?
- (c) For which $r > 0$ is $Z_n := [r^{S_n}]$ a martingale?
- (d) Is S_n a submartingale?

Of course the answers will depend on p .

Extremes

Most computations about extremes depend in one way or another on the elementary limit “ $(1 + z/n)^n \rightarrow e^z$ ” — the trick is arranging for $\mathbf{P}[X_n^* \leq x]$ to look like $(1 + z/n)^n$.

4. Let $\{X_j\}_{j \in \mathbb{N}} \stackrel{\text{iid}}{\sim} \mathbf{Ex}(\lambda)$ be independent exponentially-distributed random variables, and let

$$X_n^* := \max\{X_j : 1 \leq j \leq n\}$$

be the maximum of the first n . Find sequences $\{a_n, b_n\}$ of real numbers such that

$$Y_n := \frac{X_n^* - b_n}{a_n} \Rightarrow \mathbf{Gu}(0, 1),$$

the standard Gumbel distribution. The $\mathbf{Gu}(m, s)$ distribution has CDF

$$F(x) = \exp(-e^{-(x-m)/s})$$

for $x \in \mathbb{R}$.

5. The standard Student's t distribution with ν degrees of freedom, t_ν , has pdf

$$f_\nu(t) = \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi} \Gamma(\frac{\nu}{2})} (1 + t^2/\nu)^{-(\nu+1)/2}$$

for $t \in \mathbb{R}$. Let $\{T_j\} \stackrel{\text{iid}}{\sim} t_\nu$ and set $T_n^* := \max\{T_j : 1 \leq j \leq n\}$, and find sequences $\{a_n, b_n\}$ of real numbers such that $(T_n^* - b_n)/a_n$ converges in distribution to some limit. What's the limiting distribution? [Hints: (1) Don't get mesmerized by the normalizing constant, and (2) if x is huge and p is any fixed power then $(x+1)^p$ and $(x)^p$ don't differ by much]