

# STA 711: Probability and Measure Theory

## Analysis & Calculus Quiz

Students in STA 711: Probability & Measure Theory are expected to be familiar with real analysis at an advanced undergraduate level—the level of W. Rudin’s *Principles of Mathematical Analysis* or M. Reed’s *Fundamental Ideas of Analysis*. They should be able to answer the questions in this quiz without consulting reference materials.

**Problem 1:** Recall that a sequence  $\{x_n\}$  in a metric space  $(\mathcal{X}, d)$  *converges* to a limit  $x^* \in \mathcal{X}$  if for each  $\epsilon > 0$  there exists a number  $N_\epsilon < \infty$  such that

$$(\forall n \geq N_\epsilon) \quad d(x_n, x^*) < \epsilon.$$

a. Prove<sup>1</sup> that  $x_n := 1/\sqrt{n}$  converges to  $x^* = 0$  in the metric space  $\mathcal{X} = \mathbb{R}$  with the usual (Euclidean) distance metric  $d(x, y) := |x - y| = \sqrt{(x - y)^2}$ .

b. Find an explicit sequence  $x_n$  of rational numbers that converges to  $x^* = \pi$  in the metric space  $\mathcal{X} = \mathbb{R}$ . Prove that it converges, by finding  $N_\epsilon$  (Hint: you might want to *start* by choosing  $N_\epsilon$ — say,  $\lceil 1/\epsilon \rceil$  or  $\lceil -\log_2 \epsilon \rceil$  or  $\lceil -\log_{10} \epsilon \rceil$ — and then find  $x_n$ ).

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<sup>1</sup>Find  $N_\epsilon$  explicitly. You may find the function  $\lfloor x \rfloor := \max\{k \in \mathbb{Z} : k \leq x\}$  (the greatest integer less than or equal to  $x$ ) to be useful, or perhaps  $\lceil x \rceil := \min\{k \in \mathbb{Z} : k \geq x\}$ .

**Problem 2:** Recall that a subset  $E$  of a metric space  $(\mathcal{X}, d)$  is *open* if for each  $x \in E$  there exists some  $\epsilon_x > 0$  such that the entire ball

$$B_\epsilon(x) = \{\xi \in \mathcal{X} : d(x, \xi) < \epsilon_x\} \subset E$$

and that a set  $F \subset \mathcal{X}$  is *closed* if its complement  $F^c = \{x \in \mathcal{X} : x \notin F\}$  is open.

a. Prove that  $(0, 1)$  is open in  $\mathcal{X} = \mathbb{R}$ .

b. Prove that any union  $U = \cup E_\alpha$  of open sets is also open.

c. Show by example that the union  $U = \cup F_\alpha$  of closed sets may not be closed.

**Problem 3:** Recall that a set  $K$  in a metric space  $(\mathcal{X}, d)$  is *compact*<sup>2</sup> if every open cover  $K \subset \cup_{\alpha} U_{\alpha}$  admits a finite sub-cover  $K \subset \cup_{i=1}^n U_{\alpha_i}$ , and that a function  $f(\cdot) : \mathcal{X} \rightarrow \mathcal{Y}$  from one metric space to another is *continuous* if for every open set  $U \subset \mathcal{Y}$ ,  $f^{-1}(U) := \{x : f(x) \in U\}$  is an open set in  $\mathcal{X}$ .

a. If  $K$  is a *compact* set and  $A \subset K$  is a *closed* subset, prove that  $A$  is also compact.

b. If  $f : \mathcal{X} \rightarrow \mathbb{R}$  is a *continuous* real-valued function and  $K \subset \mathcal{X}$  is compact, prove that the supremum

$$M := \sup_{x \in K} f(x)$$

is finite.

c. Show<sup>3</sup> this can fail if  $f$  is not continuous— *i.e.*, give an example of an unbounded (but finite) function  $f$  on a compact set  $K$ .

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<sup>2</sup>The Heine-Borel theorem says *in Euclidean space* any closed & bounded set is compact, but that doesn't hold in general. For example, the unit ball  $B := \{f : \int_0^1 |f(x)|^2 dx \leq 1\}$  is closed and bounded in  $L_2((0, 1])$  but is not compact, since the sequence of functions  $\{f_n(x) := \sqrt{2} \sin(n\pi x)\} \subset B$  has no limit point in  $B$ .

<sup>3</sup>Suggestion: take  $K = [0, 1]$  on  $\mathcal{X} = \mathbb{R}$ , and define  $f(x)$  by cases. What cases?

**Problem 4:**

- a. Let  $K_\alpha$  be compact for each index  $\alpha$  and suppose that each *finite* intersection  $\bigcap_{j=1}^n K_{\alpha_j} \neq \emptyset$  is non-empty. Prove that  $\bigcap_\alpha K_\alpha \neq \emptyset$ .

- b. If  $f : \mathcal{X} \rightarrow \mathbb{R}$  is real-valued and continuous with supremum  $M := \sup_{x \in K} f(x)$  on a compact set  $K \subset \mathcal{X}$ , prove that there exists some  $x^* \in K$  for which  $f(x^*) = M$ .

**Problem 5:**

a. Give an example of a closed set  $C \subset \mathbb{R}$  that is *not* compact.

b. Give an example of a set  $A \subset \mathbb{R}$  that is neither closed nor open.

c. Give an example of a set  $B \subset \mathbb{R}$  that is both closed and open.

**Problem 6:** Evaluate the sums and integrals below for *every* value of  $p \in \mathbb{R}$  (some expressions might be infinite or undefined for some values of  $p$ ):

a.  $\int_0^1 x^p dx =$

b.  $\int_0^\infty e^{-px} dx =$

c.  $\sum_{n=2}^9 p^n =$

d.  $\sum_{n=1}^\infty p^n =$

e.  $\sum_{n=7}^\infty n p^n =$

f.  $\int_0^\infty x e^{-px^2} dx =$

g.  $\int_0^x \sin(\ln u) du =$

h.  $\int_0^\pi e^{-p \cos(x)} \sin(x) dx =$

**Problem 7:** Which of the following sums and integrals converges (to a finite limit)? Why? You need not evaluate the limit.

a. T F  $\int_2^\infty \frac{\ln(e^x - 2)}{x^3 + 1} dx$  converges:

b. T F  $\sum_{n=0}^\infty \frac{3^n (n!)^2}{(2n)!}$  converges:

c. T F  $\sum_{n=1}^\infty \frac{\ln n + \sin n}{n^{3/2}}$  converges:

d. T F  $\int_0^\infty \frac{\sin x}{x^{3/2}} dx$  converges:

e. T F  $\int_0^\infty \frac{dx}{\sqrt{x} + x^2}$  converges:

f. T F  $\int_0^1 \frac{\tan x}{x^2} dx$  converges: