Final Examination

STA 711: Probability & Measure Theory

Sunday, 2014 Dec 14, 2:00 – 5:00 pm

This is a closed-book exam. You may use a sheet of prepared notes, if you wish, but you may not share materials.

If a question seems ambiguous or confusing, *please* ask me to clarify it. Unless a problem states otherwise, you must **show** your **work**. There are blank worksheets at the end of the test if you need more room for this, and also a pdf/pmf sheet.

It is to your advantage to write your solutions as clearly as possible, and to box answers I might not find.

For full credit, answers must be given in **closed form** with no unevaluated sums, integrals, maxima, or unreduced fractions. Wherever possible, **Simplify**! Good luck.

1.	/20	6.	/20
2.	/20	7.	/20
3.	/20	8.	/20
4.	/20	9.	/20
5.	/20	10.	/20
	/100		/100
Total:	/200		

Print Name: ______________________________

Problem 1: Let $\{X_n\}$ be independent random variables on some space $(\Omega, \mathcal{F}, \mathsf{P})$, not necessarily identically distributed or L_1.

a) (5) If each $X_n > 0$ *a.s*, is it possible to have $\sum_{n=1}^{\infty} X_n < \infty$ *a.s*?
◯ Yes ◯ No. If Yes, give an example; if No, say why.

b) (5) If $X_n > 0$ and $X_n \to 0$ *a.s*, does it follow that $\sum_{n=1}^{\infty} X_n < \infty$ *a.s*?
◯ Yes ◯ No. If No, give a counter-example; if Yes, say why.

c) (10) Set $S_n := \sum_{i=1}^{n} X_i$. If $\frac{1}{n} S_n \to Y$ *a.s* for some real-valued random variable Y, use Kolmogorov's zero-one law[1] to prove that Y is almost-surely constant— *i.e.*, that for some $c \in \mathbb{R}$, $\mathsf{P}[Y = c] = 1$.

[1] *Not* his SLLN, which isn't applicable here— note these $\{X_n\}$ are not iid.

Problem 2: Let $X_n \to X$ *a.s* for some $X \in L_1(\Omega, \mathcal{F}, \mathsf{P})$, and let $Y \in L_2(\Omega, \mathcal{F}, \mathsf{P})$. For each part below answer "Yes" or "No" (4pts each).

If *Yes*, indicate which theorem best justifies your answer by selecting **Fat**ou's Lemma, Lebesgue's **D**ominated or **M**onotone **C**onvergence **T**heorems, the **B**orel/**C**antelli lemma, **Jen**sen's Inequality, or the **Min**kowski, **Höl**der, or **Mar**kov Inequalities. No need to show work.

a) If $|X_n| \le Y$, does $X_n \to X$ in L_1? ◯ No ◯ Yes, by:
◯ Fat ◯ DCT ◯ MCT ◯ B/C ◯ Jen ◯ Min ◯ Höl ◯ Mar

b) If $Y \le X_n \nearrow X$, does $X_n \to X$ in L_1? ◯ No ◯ Yes, by:
◯ Fat ◯ DCT ◯ MCT ◯ B/C ◯ Jen ◯ Min ◯ Höl ◯ Mar

c) Is $\mathsf{E}[X] \ge \liminf \mathsf{E}[X_n]$? ◯ No ◯ Yes, by:
◯ Fat ◯ DCT ◯ MCT ◯ B/C ◯ Jen ◯ Min ◯ Höl ◯ Mar

d) If $X_n \le Y$, is $\sum \mathbf{1}_{\{X_n > n\}}$ finite *a.s*? ◯ No ◯ Yes, by:
◯ Fat ◯ DCT ◯ MCT ◯ B/C ◯ Jen ◯ Min ◯ Höl ◯ Mar

e) If $X_n \searrow X \ge Y$, does $X_n \to X$ in L_1? ◯ No ◯ Yes, by:
◯ Fat ◯ DCT ◯ MCT ◯ B/C ◯ Jen ◯ Min ◯ Höl ◯ Mar

Problem 3: Let $\Omega = \{a, b, c, d\}$ and $\mathcal{F} = 2^\Omega$ with uniform probability assignment $\mathsf{P}[E] := \#(E)/4$ for $E \in \mathcal{F}$, and consider events

$$A = \{a, b, c\} \quad B = \{b, c, d\} \quad C = \{a, d\} \quad D = \{c, d\}$$

a) (5) Which (if any) **pairs** of events from $\{A, B, C, D\}$ are independent?

b) (5) Find $\sigma(A, B)$, the smallest σ-algebra on Ω containing A and B.

c) (5) Find $\sigma(A, B, C, D)$, the smallest σ-algebra on Ω containing each of the four events.

d) (5) Find $\pi(A, B, C, D)$, the smallest π-system on Ω containing each of the four events.

Problem 4:

a) (10) Recall that the pdf, CDF, and ch.f. of the normal $\mathsf{No}(\mu, \sigma^2)$ distribution are

$$\begin{aligned}
f(x) &:= (2\pi\sigma^2)^{-1/2} \exp\big(-\frac{(x-\mu)^2}{2\sigma^2}\big) &&= \frac{1}{\sigma}\varphi\big(\frac{x-\mu}{\sigma}\big)\\
F(x) &:= \int_{-\infty}^{x} f(\xi)\, d\xi &&= \Phi\big(\frac{x-\mu}{\sigma}\big)\\
\chi(\omega) &:= \mathsf{E}\big[e^{i\omega X}\big] &&= \exp\Big(i\mu\omega - \frac{\sigma^2\omega^2}{2}\Big)
\end{aligned}$$

Let $X_n \sim \mathsf{No}(\mu_n, \sigma_n^2)$ for each $n \in \mathbb{N}$. If $\mu_n \to 1$ and $\sigma_n^2 \to 4$ as $n \to \infty$, prove that $X_n \Rightarrow \mathsf{No}(1, 2^2)$, *i.e.*, that the random variables converge in distribution to the indicated normal

b) (10) If $a_n \to 0$ is a sequence of real numbers and if $X \in L_1(\Omega, \mathcal{F}, \mathsf{P})$, prove that the sequence $Y_n := X + a_n$ converges in distribution to X.

Problem 5: Let $\{A_n\}$ be independent events on some probability space $(\Omega, \mathcal{F}, \mathsf{P})$ with $\mathsf{P}(A_n) = 1/n$ for $n \in \mathbb{N}$ and set

$$X_n := \sqrt{n}\mathbf{1}_{A_n}.$$

a) (4) Is $\{X_n\}$ uniformly bounded in L_1? ◯ Yes ◯ No Why?

b) (4) Is $\{X_n\}$ uniformly bounded in L_3? ◯ Yes ◯ No Why?

c) (4) Is $\{X_n^2\}$ Uniformly Integrable? ◯ Yes ◯ No Why?

d) (4) Find the covariance of X_n and X_m for **all** $n, m \in \mathbb{N}$.

e) (4) In which (if any) sense(s) does $X_n \to 0$ as $n \to \infty$?

◯ $a.s.$ ◯ $pr.$ ◯ L_1 ◯ L_2 ◯ L_∞ ◯ $in\ dist.$

Problem 6: Let $X, Y \overset{\text{iid}}{\sim} \mathsf{No}(0,1)$ and set $S := X + Y$, $Z := X - Y$, and $A := \{X < Y\} = \{Z < 0\}$.

a) (8) Find the indicated covariances:

$\mathsf{Cov}(S, Z) =$ ________ $\mathsf{Cov}(Y, Z) =$ ________

b) (12) Find the indicated conditional probabilities:

$\mathsf{P}[A \mid X] =$ ________ $\mathsf{P}[A \mid S] =$ ________

Problem 7: Again let $X, Y \overset{\text{iid}}{\sim} \mathsf{No}(0,1)$ and set $S := X+Y$, $Z := X-Y$, and $A := \{X < Y\} = \{Z < 0\}$.

a) (8) Find the indicated conditional expectations:

$\mathsf{E}[X \mid A] =$ ________ $\mathsf{E}[S \mid A] =$ ________

b) (6) Find the indicated conditional expectations:

$\mathsf{E}[X \mid S] =$ ________ $\mathsf{E}[Z \mid Y] =$ ________

c) (6) Find the indicated conditional expectations:

$\mathsf{E}[X \mid S, Z] =$ ________ $\mathsf{E}[Z \mid X, S] =$ ________

Problem 8: Let $\Omega = (0,1]^2 = \{\omega = (\omega_1, \omega_2) : \ 0 < \omega_j \le 1\}$ be the unit square, with the Borel sets $\mathcal{F} = \mathcal{B}(\Omega)$ and Lebesgue measure P (*i.e.*, area).

a) (5) Give an example[2] of an event $A \in \mathcal{F}$ and random variable W on $(\Omega, \mathcal{F}, \mathsf{P})$ that are independent and non-trivial— no constant RVs or null events.

b) (5) Give an example of a random variable X on $(\Omega, \mathcal{F}, \mathsf{P})$ that is almost-surely finite but not bounded, if possible; if not, explain.

c) (5) Give an example Y of a random variable in $L_1(\Omega, \mathcal{F}, \mathsf{P})$ that is not in $L_2(\Omega, \mathcal{F}, \mathsf{P})$, if possible; if not, explain.

d) (5) Give an example Z of a random variable on $(\Omega, \mathcal{F}, \mathsf{P})$ whose distribution is neither continuous nor discrete, if possible; if not, explain.

[2]You must be explicit— give $W(\omega)$ for every $\omega \in \Omega$, for example.

Problem 9: Let $\{X_n\} \overset{\text{iid}}{\sim} \mathsf{Ex}(1)$ be iid exponentially-distributed with mean 1, and set $X_n^* := \max\{X_1, ..., X_n\}$ and $S_n := \sum_{1\le j\le n} X_j$.

a) (5) Show that $\mathsf{P}\big[X_n^* \le (z + \log n)\big]$ converges as $n \to \infty$ and find the limiting value $G(z) := \lim_{n\to\infty} \mathsf{P}\big[X_n^* \le (z + \log n)\big]$ for every $z \in \mathbb{R}$.

b) (5) Find the approximate median M_n, so $\mathsf{P}[X_n^* \le M_n] \approx 1/2$.

c) (5) In which sense(s) does $S_n/n \to 1$?

○ $a.s.$ ○ $pr.$ ○ L_1 ○ L_2 ○ L_∞ ○ $in\ dist.$

d) (5) Find the smallest bounds $a_n > 0$ and $b_n > 0$ you can to ensure

$$X_n^* \le a_n S_n \qquad S_n \le b_n X_n^*$$

$a_n =$ ______________ $b_n =$ ______________

Problem 10: Let $\{X, Y, Z\}$ be three RVs on the same probability space $(\Omega, \mathcal{F}, \mathsf{P})$. Choose True or False below; no need to explain (unless you can't resist). Each is 2pt.

a) T F If X is independent of Y and also X is independent of Z then X is independent of $Y + Z$.

b) T F If $\mathsf{P}[X < Y < Z] = 1$ then it's not possible for X, Y, Z to be independent.

c) T F If X, Y, Z are independent and all are L_1 then the product XYZ is L_1 too.

d) T F If Z is measurable over $\sigma(X, Y)$ then there exists some Borel function g on $\mathbb{R}^2$ such that $Z = g(X, Y)$ *a.s*

e) T F If X, Y are independent then the events $A := [X < 0]$ and $B := [Y > 0]$ are independent too.

f) T F If $\sigma(X) \subset \sigma(Y)$ and $X, Y \in L_1$ then $\mathsf{E}[X \mid Y]$ is the constant random variable with value $\mathsf{E}[X]$.

g) T F If only finitely-many of the events $A_n := [X > n]$ occur a.s., then $X \in L_1(\Omega, \mathcal{F}, \mathsf{P})$.

h) T F If $X, Y \in L_2$ then $XY \in L_1$ with $\|XY\|_1 \le \|X\|_2 \|Y\|_2$.

i) T F Let $\phi(\omega) := \mathsf{E}[e^{i\omega X}]$ and $\psi(\omega) := \mathsf{E}[e^{i\omega Y}]$ be the ch.f.s for X and Y. Then the ch.f. for $Z := X/Y$ is $\phi(\omega)/\psi(\omega)$ if X, Y are independent.

j) T F If $X \in L_1$, $Y \in L_2$, and $Z \in L_3$, then $(X + Y + Z) \in L_2$.

Blank Worksheet

Another Blank Worksheet

Name	Notation	pdf/pmf	Range	Mean μ	Variance σ^2	
Beta	$\mathsf{Be}(\alpha,\beta)$	$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$x\in(0,1)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	
Binomial	$\mathsf{Bi}(n,p)$	$f(x)=\binom{n}{x}p^x q^{(n-x)}$	$x\in 0,\cdots,n$	$n\,p$	$n\,p\,q$	$(q=1-p)$
Exponential	$\mathsf{Ex}(\lambda)$	$f(x)=\lambda\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	$1/\lambda$	$1/\lambda^2$	
Gamma	$\mathsf{Ga}(\alpha,\lambda)$	$f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	α/λ	α/λ^2	
Geometric	$\mathsf{Ge}(p)$	$f(x)=p\,q^x$	$x\in\mathbb{Z}_+$	q/p	q/p^2	$(q=1-p)$
		$f(y)=p\,q^{y-1}$	$y\in\{1,...\}$	$1/p$	q/p^2	$(y=x+1)$
HyperGeo.	$\mathsf{HG}(n,A,B)$	$f(x)=\frac{\binom{A}{x}\binom{B}{n-x}}{\binom{A+B}{n}}$	$x\in 0,\cdots,n$	$n\,P$	$n\,P\,(1-P)\frac{N-n}{N-1}$	$(P=\frac{A}{A+B})$
Logistic	$\mathsf{Lo}(\mu,\beta)$	$f(x)=\frac{e^{-(x-\mu)/\beta}}{\beta[1+e^{-(x-\mu)/\beta}]^2}$	$x\in\mathbb{R}$	μ	$\pi^2\beta^2/3$	
Log Normal	$\mathsf{LN}(\mu,\sigma^2)$	$f(x)=\frac{1}{x\sqrt{2\pi\sigma^2}}e^{-(\log x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}_+$	$e^{\mu+\sigma^2/2}$	$e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right)$	
Neg. Binom.	$\mathsf{NB}(\alpha,p)$	$f(x)=\binom{x+\alpha-1}{x}p^\alpha\,q^x$	$x\in\mathbb{Z}_+$	$\alpha q/p$	$\alpha q/p^2$	$(q=1-p)$
		$f(y)=\binom{y-1}{y-\alpha}p^\alpha\,q^{y-\alpha}$	$y\in\{\alpha,...\}$	α/p	$\alpha q/p^2$	$(y=x+\alpha)$
Normal	$\mathsf{No}(\mu,\sigma^2)$	$f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}$	μ	σ^2	
Pareto	$\mathsf{Pa}(\alpha,\epsilon)$	$f(x)=\alpha\,\epsilon^\alpha/x^{\alpha+1}$	$x\in(\epsilon,\infty)$	$\frac{\epsilon\,\alpha}{\alpha-1}$	$\frac{\epsilon^2\alpha}{(\alpha-1)^2(\alpha-2)}$	
Poisson	$\mathsf{Po}(\lambda)$	$f(x)=\frac{\lambda^x}{x!}e^{-\lambda}$	$x\in\mathbb{Z}_+$	λ	λ	
Snedecor F	$F(\nu_1,\nu_2)$	$f(x)=\frac{\Gamma(\frac{\nu_1+\nu_2}{2})(\nu_1/\nu_2)^{\nu_1/2}}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})}\times$ $x^{\frac{\nu_1-2}{2}}\left[1+\frac{\nu_1}{\nu_2}x\right]^{-\frac{\nu_1+\nu_2}{2}}$	$x\in\mathbb{R}_+$	$\frac{\nu_2}{\nu_2-2}$	$\left(\frac{\nu_2}{\nu_2-2}\right)^2\frac{2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)}$	
Student t	$t(\nu)$	$f(x)=\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}}[1+x^2/\nu]^{-(\nu+1)/2}$	$x\in\mathbb{R}$	0	$\nu/(\nu-2)$	
Uniform	$\mathsf{Un}(a,b)$	$f(x)=\frac{1}{b-a}$	$x\in(a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	
Weibull	$\mathsf{We}(\alpha,\beta)$	$f(x)=\alpha\beta\,x^{\alpha-1}\,e^{-\beta\,x^\alpha}$	$x\in\mathbb{R}_+$	$\frac{\Gamma(1+\alpha^{-1})}{\beta^{1/\alpha}}$	$\frac{\Gamma(1+2/\alpha)-\Gamma^2(1+1/\alpha)}{\beta^{2/\alpha}}$	