

## Sta 711: Homework 2

### $\sigma$ -Algebras and partitions.

Fields and  $\sigma$ -fields generated by *partitions* (finite or countable collections of nonempty *disjoint* events  $\Lambda_j \in \mathcal{F}$  with  $\cup \Lambda_j = \Omega$ ), and probability assignments on them, are especially easy to describe. Let  $\{A, B\} \subset \mathcal{F}$  be two events in a probability space  $(\Omega, \mathcal{F}, \mathbf{P})$ , not necessarily nonempty or disjoint. Let  $\mathcal{P} = \mathcal{P}(A, B)$  be the partition generated by these events, *i.e.*, the smallest partition for which  $\{A, B\} \subset \sigma(\mathcal{P})$ .

1. Enumerate all possible elements of the partition  $\mathcal{P}$ . How many distinct nonempty elements does  $\mathcal{P}$  have, at most? How many, at minimum?
2. How many distinct elements does the  $\sigma$ -algebra  $\sigma(\mathcal{P})$  contain, at most? At minimum? Describe them in words (don't list them, there are too many).

### Null sets.

3. Let  $\{A_n, n \in \mathbb{N}\}$  be events with  $\mathbf{P}(A_n) = 1$ . Prove that  $\mathbf{P}(\cap_{n=1}^{\infty} A_n) = 1$ .
4. Now consider uncountably many events  $\{B_\alpha\}$ , all with  $\mathbf{P}(B_\alpha) = 1$ . Does it follow necessarily that  $\mathbf{P}(\cap_\alpha B_\alpha) = 1$ ? Give a proof or a counter example.
5. Let  $n \in \mathbb{N}$  and let  $\{C_k\}$  be a collection of  $n$  events such that  $\sum_{k=1}^n \mathbf{P}(C_k) > n - 1$ . Show that  $\mathbf{P}(\cap_{k=1}^n C_k) > 0$ .

### Distribution functions and continuity.

6. Give an example<sup>1</sup> of a real-valued function on  $\mathbb{R}$  which is continuous, but **not** uniformly continuous.
7. Let  $G$  be a continuous distribution function on  $\mathbb{R}$ . Show<sup>2</sup> that  $G$  is in fact *uniformly* continuous, *i.e.*, that  $(\forall \epsilon > 0)(\exists \delta > 0)(\forall x, y \in \mathbb{R}) |x - y| < \delta \Rightarrow |G(x) - G(y)| < \epsilon$ .
8. Show that any distribution function  $F$  on  $\mathbb{R}$  can have *at most countably many* discontinuities. Hint: Consider the open intervals  $(F(x-), F(x))$  for discontinuity points  $x$ .
9. Let  $\{A_n\}_{n \in \mathbb{N}} \subset \mathcal{F}$  be *increasing* in the sense that each  $A_n \subset A_{n+1}$ . Prove that  $\mathbf{P}(A_n) \rightarrow \mathbf{P}(\cup_{n \in \mathbb{N}} A_n)$ , a property called “continuity”. What happens for  $\{B_n\} \subset \mathcal{F}$  with  $B_n \supset B_{n+1}$ ?

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<sup>1</sup>and, of course, *prove* that your example satisfies the criteria.

<sup>2</sup>Hint: Consider points  $\{x_i\}$  for which  $G(x_i) = i/n$  for  $1 \leq i < n$ . Must these exist? If they do, are they determined uniquely? Does that matter? Draw a graph!

