

Sta 711: Homework 8

Convergence Of Series, Strong Law

1. For $n \in \mathbb{N}$ let $j = \lfloor \log_2 n \rfloor$ and $i = n - 2^j$, so $n = i + 2^j$ with $j \in \mathbb{N}_0$ and $0 \leq i < 2^j$. For $\omega \in \Omega = (0, 1]$ set

$$X_n(\omega) := n \mathbf{1}_{\{i/2^j < \omega \leq (i+1)/2^j\}}$$

Verify that X_n converges *pr.* but not *a.s.* (for Lebesgue measure \mathbf{P} on the Borel sets \mathcal{F}), and find an explicit subsequence that converges almost-surely. What is the limit? Does this subsequence converge in L_1 ?

2. One version of the SLLN states that if $\{X_n, n \geq 1\}$ are iid with $\mathbf{E}|X_1| < \infty$, then $S_n/n \rightarrow \mathbf{E}(X_1)$ a.s. Show that also

$$S_n/n \rightarrow \mathbf{E}(X_1) \quad \text{in } L_1.$$

3. Define a sequence $\{X_n\}$ of random variables iteratively as follows. Let $X_0 \equiv c > 0$ be any positive constant and, for $n \in \mathbb{N}$, let X_n have a uniform distribution on $(0, X_{n-1}]$ (independent of $\{X_j : j < n - 1\}$). Show that

$$\frac{1}{n} \log X_n$$

converges a.s. and find the almost sure limit.

Two Fun Concepts

4. Let f_0 and f_1 be probability mass functions (pmfs) on the set $\mathcal{S} := \{1, 2, \dots, 100\}$, i.e., nonnegative functions satisfying $\sum_{y \in \mathcal{S}} f_\theta(y) = 1$ for $\theta = 0, 1$. Let $\{X_n\}$ be iid random variables with pmf f_0 , so $\mathbf{P}[X_n = y] = f_0(y)$ for $y \in \mathcal{S}$. Set

$$\Lambda_n := \prod_{i=1}^n \frac{f_1(X_i)}{f_0(X_i)}$$

Prove that $\Lambda_n \rightarrow 0$ almost surely if $f_0(y) \neq f_1(y)$ for at least one $y \in \mathcal{S}$. Be careful about any points where $f_0(y) = 0$ or $f_1(y) = 0$. The quantity Λ_n is called the *Bayes factor* or *likelihood ratio* against the “null hypothesis” f_0 . This shows that the Likelihood Ratio Test always succeeds (eventually!).¹

5. Suppose $g : \mathbb{R}_+ \mapsto \mathbb{R}$ is measurable and Lebesgue integrable. Let $\{X_n, n \geq 1\} \stackrel{\text{iid}}{\sim} \text{Ex}(1)$ be standard exponential random variables with pdf $f(x) = e^{-x} \mathbf{1}_{\{x > 0\}}$ and define $Y_n := g(X_n) \exp(X_n)$. What is the limit as $n \rightarrow \infty$ of $\bar{Y}_n = \sum_{i=1}^n Y_i/n$? In what sense, and why? If $g \in L_2(\mathbb{R}_+, e^x dx)$, find the variance of \bar{Y}_n and show that it converges to zero as $n \rightarrow \infty$. Show how this lets us estimate $\int_{\mathbb{R}_+} g(x) dx$.

¹Hint: $(\forall x \in \mathbb{R}) e^x \geq 1 + x$ (with equality only at $x = 0$), and so $(\forall y > 0) \log y \leq y - 1$. Or, for another approach, apply Jensen’s inequality to the convex function $-\log y$ on \mathbb{R}_+ .