Sta 711: Homework 11

Martingales

A sequence $\{(X_n, \mathcal{F}_n),\ n \geq 0\}$ of random variables $X_n \in L_1(\Omega, \mathcal{F}, \mathsf{P})$ and a nested sequence of σ-algebras $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}$ is a *martingale* if for each $0 \leq n \leq m < \infty$

$$X_n = \mathsf{E}[X_m \mid \mathcal{F}_n],$$

so on average the sequence neither increases nor decreases. Note this implies that X_n is $\mathcal{F}_n$-measurable. By the tower property of conditional expectations, it is enough to check this condition for $m = n + 1$.

1. A sequence $\{(X_n, \mathcal{F}_n),\ n \geq 0\}$ is *predictable* if X_{n+1} is $\mathcal{F}_n$-measurable for each n. Show every predictable martingale is constant (*i.e.*, $X_n = X_0$ a.s.).

2. A sequence $\{(X_n, \mathcal{F}_n),\ n \geq 0\} \subset L_1(\Omega, \mathcal{F}, \mathsf{P})$ is a *submartingale* if, for each $0 \leq n \leq m < \infty$, $\mathcal{F}_n \subseteq \mathcal{F}_m \subseteq \mathcal{F}$ and $X_n \leq \mathsf{E}[X_m \mid \mathcal{F}_n]$— so, on average, X_n is increasing.

   Let $\{(X_n, \mathcal{F}_n),\ n \geq 0\}$ and $\{(Y_n, \mathcal{F}_n),\ n \geq 0\}$ be submartingales on $(\Omega, \mathcal{F}, \mathsf{P})$. Show that their max $(X_n \vee Y_n)$ and sum $(X_n + Y_n)$ are submartingales too.

3. Show that every submartingale $\{(X_n, \mathcal{F}_n)\}$ can be written as the sum $X_n = (M_n + A_n)$ of a martingale $\{(M_n, \mathcal{F}_n)\}$ and a predictable non-decreasing process A_n, and that the decomposition is unique if we set $A_0 = 0$. Suggestion: How must A_n be defined to make $A_0 = 0$ and $\mathsf{E}[(X_{n+1} - A_{n+1}) \mid \mathcal{F}_n] = (X_n - A_n)$?

4. Let $\{(M_n, \mathcal{F}_n)\}$ be a martingale and $\phi : \mathbb{R} \to \mathbb{R}$ a convex function for which $X_n := \phi(M_n)$ is in $L_1(\Omega, \mathcal{F}, \mathsf{P})$. Show that $\{(X_n, \mathcal{F}_n)\}$ is a submartingale.

5. Fix $0 < p < 1$, set $q := 1 - p$, and let $\{\xi_j\}$ be iid random variables with $\mathsf{P}[\xi_j = 1] = p$ and $\mathsf{P}[\xi_j = -1] = q$. Set

   $$S_n := \sum_{j \leq n} \xi_j,$$

   a random walk (possibly an asymmetric one) on the integers starting at $S_0 = 0$.

   (a) For which $\alpha \in \mathbb{R}$ is $X_n := [S_n - \alpha n]$ a martingale?

   (b) For which $\alpha, \beta \in \mathbb{R}$ is $Y_n := [(S_n - \alpha n)^2 - \beta n]$ a martingale?

   (c) For which $r > 0$ is $Z_n := [r^{S_n}]$ a martingale?

   (d) Is S_n a submartingale?

   Of course the answers will depend on p.

Extremes

Most computations about extremes depend in one way or another on the elementary limit "$(1+z/n)^n \to e^z$"— the trick is arranging for $\mathsf{P}[X_n^* \le x]$ to look like $(1+z/n)^n$.

6. Let $\{X_j\}_{j\in\mathbb{N}} \overset{\text{iid}}{\sim} \mathsf{Ex}(\lambda)$ be independent exponentially-distributed random variables, and let

$$X_n^* := \max\{X_j :\ 1 \le j \le n\}$$

be the maximum of the first n. Find sequences $\{a_n, b_n\}$ of real numbers such that

$$Y_n := \frac{X_n^* - b_n}{a_n} \Rightarrow \mathsf{Gu}(0,1),$$

the standard Gumbel distribution. The $\mathsf{Gu}(m,s)$ distribution has CDF

$$F(x) = \exp\big(-e^{-(x-m)/s}\big)$$

for $x \in \mathbb{R}$. The maxima from many other distributions (normal, gamma, etc.) are also approximately Gumbel (you don't have to prove that!).

7. The standard Student's t distribution with ν degrees of freedom, t_ν, has pdf

$$f_\nu(t) = \frac{\Gamma\big(\frac{\nu+1}{2}\big)}{\sqrt{\nu\pi}\,\Gamma\big(\frac{\nu}{2}\big)}\ (1+t^2/\nu)^{-(\nu+1)/2}$$

for $t \in \mathbb{R}$. Let $\{T_j\} \overset{\text{iid}}{\sim} t_\nu$ and set $T_n^* := \max\{T_j :\ 1 \le j \le n\}$, and find sequences $\{a_n, b_n\}$ of real numbers such that $(T_n^* - b_n)/a_n$ converges in distribution to some limit. What's the limiting distribution? You don't need to prove it, but this is also the limiting distribution for the maxima of many other heavy-tailed distributions (Pareto, α-Stable, log Normal, *etc.*).

Hints: (1) Don't get mesmerized by the normalizing constant, and (2) if x is huge and p is any fixed power then $(x+1)^{-p}$ and $(x)^{-p}$ don't differ by much