December 12, 2017

711 class,

Yesterday in the Review Session I punted on answering Problem 7 on the 2016 Final Exam. Simplifying a little (by conditioning on the event $Z \neq 0$), the problem is:

$$\text{Let } X, Y \overset{\text{iid}}{\sim} \mathsf{No}(0,1) \text{ and set } Z := 3X + 4Y. \text{ Find } \mathsf{E}[X \mid Z].$$

The joint distribution of X and Z is bivariate Gaussian, so (as the footnote hint suggests) the solution *must* be of the form

$$\mathsf{E}[X \mid Z] = a + bZ$$

for some $a, b \in \mathbb{R}$. We can find a, b by projection or, equivalently, by using the fact that

$$\mathsf{E}\Big[\mathsf{E}[X \mid Z] \cdot g(Z)\Big] = \mathsf{E}\Big[X \cdot g(Z)\Big]$$

for any Borel indicator function $g(\cdot)$ (that's the definition of conditional expectation) or, by linearity and LDCT, any Borel $g(\cdot)$ for which $X \cdot g(Z) \in L_1$— in particular, for $g(z) := 1$ and $g(z) := z$:

$$\begin{aligned}
\mathsf{E}\big[(a + bZ) \cdot 1\big] &= \mathsf{E}\big[X \cdot 1\big] \\
&= 0, \qquad \text{so} \\
a &= 0. \\
\mathsf{E}\big[(a + bZ) \cdot Z\big] &= \mathsf{E}\big[X \cdot Z\big] \\
&= \mathsf{E}\big[X(3X + 4Y) \\
&= 3, \qquad \text{so} \\
b\ \mathsf{E}[Z^2] &= 3 \\
b &= 3/25
\end{aligned}$$

since $\mathsf{E}[X^2] = 1$, $\mathsf{E}[XY] = 0$, and $\mathsf{E}[Z^2] = 3^2 + 4^2 = 25$. Thus

$$\mathsf{E}[X \mid Z] = (3Z/25) = 0.12Z.$$

If we had $Z = X_1 + X_2 + X_3 \quad + \quad Y_1 + Y_2 + Y_3 + Y_4$ with all the $\{X_i, Y_j\} \overset{\text{iid}}{\sim} \mathsf{No}(0,1)$, *then* each $\mathsf{E}[X_i \mid Z]$ and $\mathsf{E}[Y_j \mid Z]$ would be $Z/7$ by symmetry, making it tempting to guess in Problem 7 that $\mathsf{E}[X \mid Z]$ is $Z/7$ (or maybe $3Z/7$ or $4Z/7$), but we have to resist some temptations, the answer is $(3/25)Z$.

Also— after class some people asked me about Problem 2c) on the 2015 Final exam:

2c) Let $X_n \to X$ *pr.* and $Y \in L_2$. If $X_n \le Y$, is $\mathsf{E}[X] \ge \limsup \mathsf{E}[X_n]$?

If we had $X_n \to X$ *almost surely*, then we would have $X = \limsup X_n$ and $Y \in L_2 \subset L_1$ so the result would be "yes" by Corollary 2 to Fatou's Lemma in the Week 4 class notes: Since $X_n \le Y \in L_1$, Fatou's Lemma says the nonnegative RVs $Y_n := [Y - X_n] \ge 0$ satisfy

$$\begin{aligned} \liminf \mathsf{E}[Y_n] &\ge \mathsf{E}\big[\liminf Y_n\big], \qquad \textit{i.e.,} \\ \mathsf{E}[Y] - \limsup \mathsf{E}[X_n] &\ge \mathsf{E}[Y] - \mathsf{E}\big[\limsup X_n\big], \end{aligned}$$

so

$$\mathsf{E}[X] = \mathsf{E}\big[\limsup X_n\big] \ge \limsup \mathsf{E}[X_n].$$

BUT, because we only have convergence *pr.*, the relation "$X \stackrel{?}{=} \limsup X_n$" may fail and we need to do a little more work.

Suppose, for conradiction, that $\mathsf{E}[X] < \limsup \mathsf{E}[X_n]$. Then for some $\epsilon > 0$ and some subsequence $n_i \to \infty$,

$$\mathsf{E}[X] + \epsilon \le \mathsf{E}\big[X_{n_i}\big]$$

for each $i \in \mathbb{N}$. Now, since $X_{n_i} \to X$ (*pr.*), find a further subsequence n_{i_j} along which $X_{n_{i_j}} \to X$ (*a.s.*). But $X_{n_{i_j}} \le Y \in L_1$ and $X = \limsup X_{n_{i_j}}$ almost surely, so Corollary 2 to Fatou's lemma says

$$\mathsf{E}[X] = \mathsf{E}\big[\limsup X_{n_{i_j}}\big] \ge \limsup \mathsf{E}\big[X_{n_{i_j}}\big] \ge \mathsf{E}[X] + \epsilon,$$

a contradiction. Thanks to Shuxi Zeng for spotting a gap in an earlier version of this note.

-RLW