Transformations and Normality

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STA721 Linear Models

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November 28, 2017

Outline

Topics

- Normality & Transformations
- Box-Cox
- Nonlinear Regression

Readings: Christensen Chapter 13 & Wakefield Chapter 6

Linear Model

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- Normal Distribution for $\boldsymbol{\epsilon}$ with constant variance
- Outlier Models
- Robustify with heavy tailed error distributions
- Computational Advantages of Normal Models

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[1] independent but not identically distributed

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"Supernormality of residuals"

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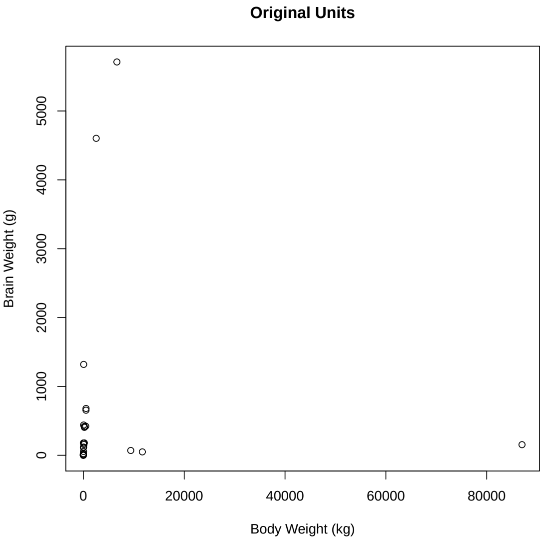
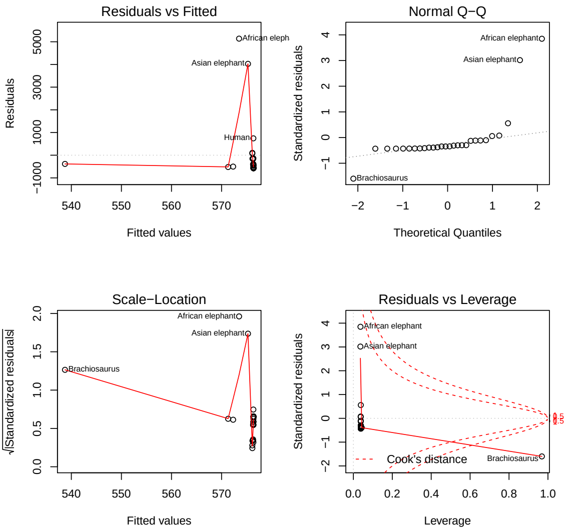
- Judgment call - use simulations to gain experience!

Animal Example



Residual Plots



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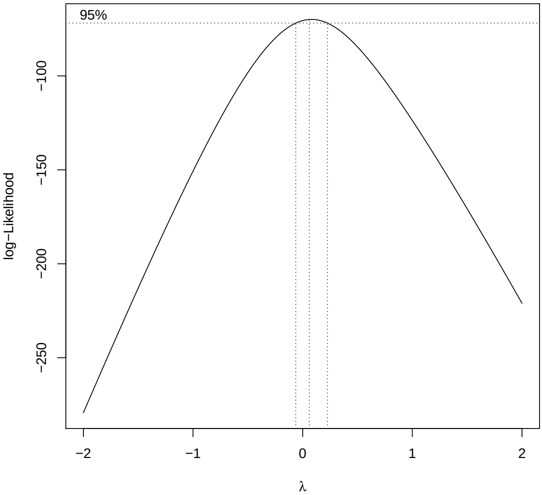
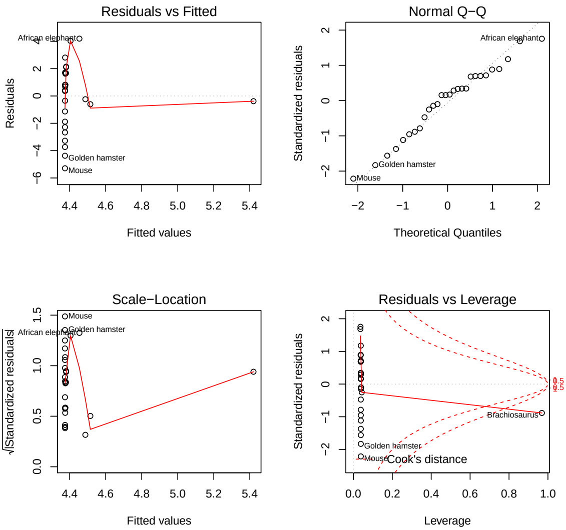
- Jacobian term is $\prod_i y_i^{\lambda-1}$ for all λ
- Profile Likelihood based on substituting MLE $\boldsymbol{\beta}$ and σ^2 for each value of λ is

$$\log(\mathcal{L}(\lambda) \propto (\lambda - 1) \sum_i \log(Y_i) - \frac{n}{2} \log(\mathrm{SSE}(\lambda))$$

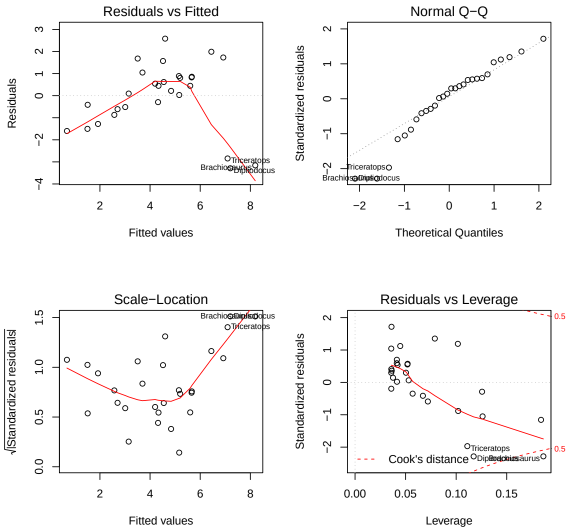
Profile Likelihood



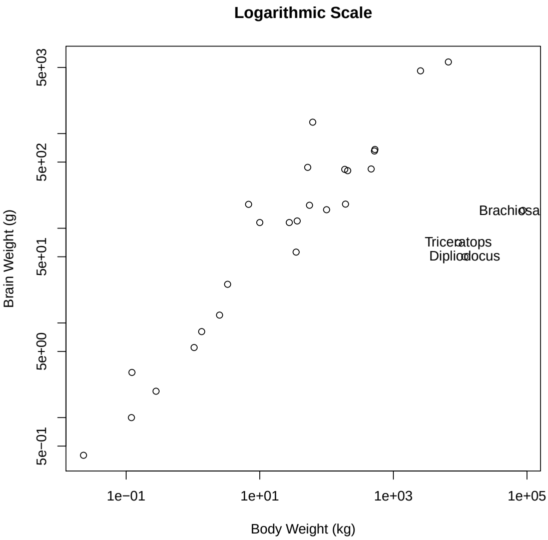
Residuals After Transformation of Response



Residuals After Transformation of Both



Transformed Data



Test that Dinos are Outliers

	Res.Df	RSS	Df	Sum of Sq	F	Pr(>F)
1	23	12.12				
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log(body)	0.7523	0.0457	16.45	0.0000
Triceratops	-4.7839	0.7913	-6.05	0.0000
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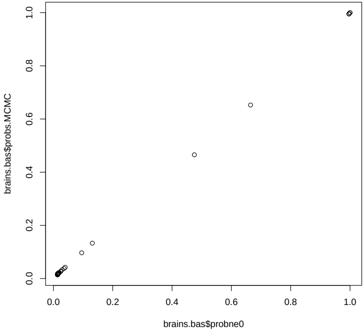
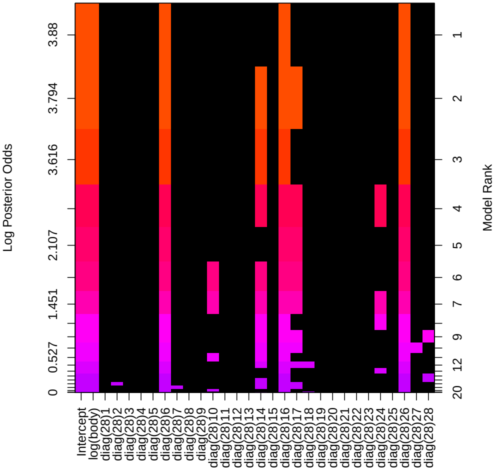
Dinosaurs come from a different population from mammals

Model Selection Priors

```
brains.bas = bas.lm(log(brain) ~ log(body) + diag(28),
   data=Animals,  prior="hyper-g-n", a=3,
   modelprior=beta.binomial(1,28),
   method="MCMC", n.models=2^17, MCMC.it=2^18)
# check for convergence
plot(brains.bas$probne0, brains.bas$probs.MCMC)
```



image(brains.bas)




```
rownames(Animals)[c(6, 14, 16, 26)]
"Dipliodocus" "Human" "Triceratops" "Brachiosaurus"
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Generalized Linear Models preferable

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\log(\mu(\boldsymbol{\beta})) &= \log\left[\frac{D}{V}\exp(-\kappa_e x_i)\right] \\
&= \log[D] - \log(V) - \kappa_e x_i \\
log(Y_i) - \log[30] &= \beta_0 + \beta_1 x_i + \epsilon_i
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Nonlinear Least Squares

```
> conc.nlm = nls( log(y) ~ log((30/V)*exp(-k*x)),
                  data=df, start=list(V=vhat, k=khat))
> summary(conc.nlm)
Formula: log(y) ~ log((30/V) * exp(-k * x))
Parameters:
                Estimate Std. Error t value Pr(>|t|)
V.(Intercept) 16.66331    7.11923   2.341 0.057796 .
k.x            0.15211    0.02368   6.423 0.000673 ***

Residual standard error: 0.7411 on 6 degrees of freedom
Number of iterations to convergence: 0
Achieved convergence tolerance: 3.978e-09
```

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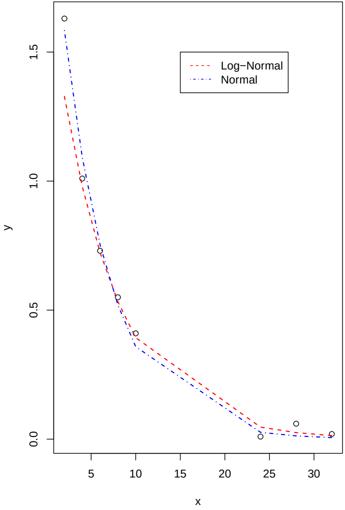
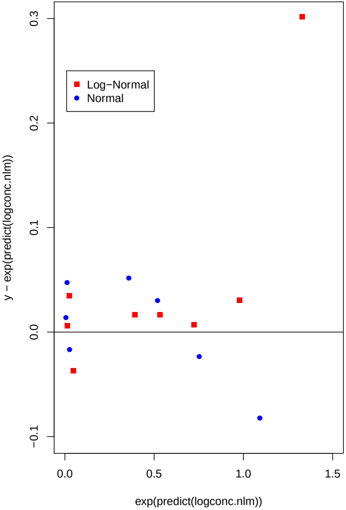
Intrinsically Nonlinear Model

```
> summary(conc.nlm)
Formula: y ~ (30/V) * exp(-k * x)
Parameters:
  Estimate Std. Error t value Pr(>|t|)
V 13.06506    0.60899   21.45 6.69e-07 ***
k  0.18572    0.01124   16.52 3.14e-06 ***
---
Residual standard error: 0.05126 on 6 degrees of freedom
Number of iterations to convergence: 4
Achieved convergence tolerance: 7.698e-06
```

Fitted Values & Residuals





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- Asymptotic Distributions

Bayes obtain the posterior directly for parameters and functions of parameters! Priors? Constraints on Distributions?

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- Nonlinear Models as suggested by Theory or Generalized Linear Models are alternatives
- "normal" estimates may be useful approximations for large p or for starting values for more complex models (where convergence may be sensitive to starting values)