Choice of Prior Distributions

STA721 Linear Models Duke University

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Bayesian Estimation

Model

$$\mathbf{Y} \sim \mathrm{N}(\mathbf{X}\boldsymbol{\beta}, \mathbf{I}_n/\phi)$$

precision $\phi = 1/\sigma^2$

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\Phi_n &= \mathbf{X}^T\mathbf{X} + \Phi_0 \\
\mathbf{b}_n &= \Phi_n^{-1}(\mathbf{X}^T\mathbf{X}\hat{\boldsymbol{\beta}} + \Phi_0\mathbf{b}_0) \\
\nu_n &= \nu_0 + n \\
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\end{aligned}$$

Marginal Distribution from Normal–Gamma

Theorem

Let $\boldsymbol{\theta} \mid \phi \sim N(m, \frac{1}{\phi}\Sigma)$ *and* $\phi \sim \mathbf{G}(\nu/2, \nu\hat{\sigma}^2/2)$. *Then* $\boldsymbol{\theta}$ $(p \times 1)$ *has a p dimensional multivariate t distribution*

$$\boldsymbol{\theta} \sim t_\nu(m, \hat{\sigma}^2\Sigma)$$

with density

$$p(\boldsymbol{\theta}) \propto \left[1 + \frac{1}{\nu}\frac{(\boldsymbol{\theta} - m)^T\Sigma^{-1}(\boldsymbol{\theta} - m)}{\hat{\sigma}^2}\right]^{-\frac{p+\nu}{2}}$$

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Any linear combination $\lambda^T\boldsymbol{\beta}$

$$\lambda^T\boldsymbol{\beta} \mid \mathbf{Y} \sim t_{\nu_n}(\lambda^T\mathbf{b}_n, \hat{\sigma}^2\lambda^T\Phi_n^{-1}\lambda)$$

has a univariate t distribution with $\mathbf{v}_n$ degrees of freedom

Predictive Distribution

Suppose $\mathbf{Y}^* \mid \boldsymbol{\beta}, \phi \sim \mathsf{N}(\mathbf{X}^*\boldsymbol{\beta}, \mathbf{I}/\phi)$ and is conditionally independent of $\mathbf{Y}$ given $\boldsymbol{\beta}$ and ϕ

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A class of prior distributions $\mathcal{P}$ for $\boldsymbol{\theta}$ is conjugate for a sampling model $p(y \mid \boldsymbol{\theta})$ if for every $p(\boldsymbol{\theta}) \in \mathcal{P}$, $p(\boldsymbol{\theta} \mid \mathbf{Y}) \in \mathcal{P}$.

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Cannot represent real prior beliefs; double use of data but has the "right" behaviour.

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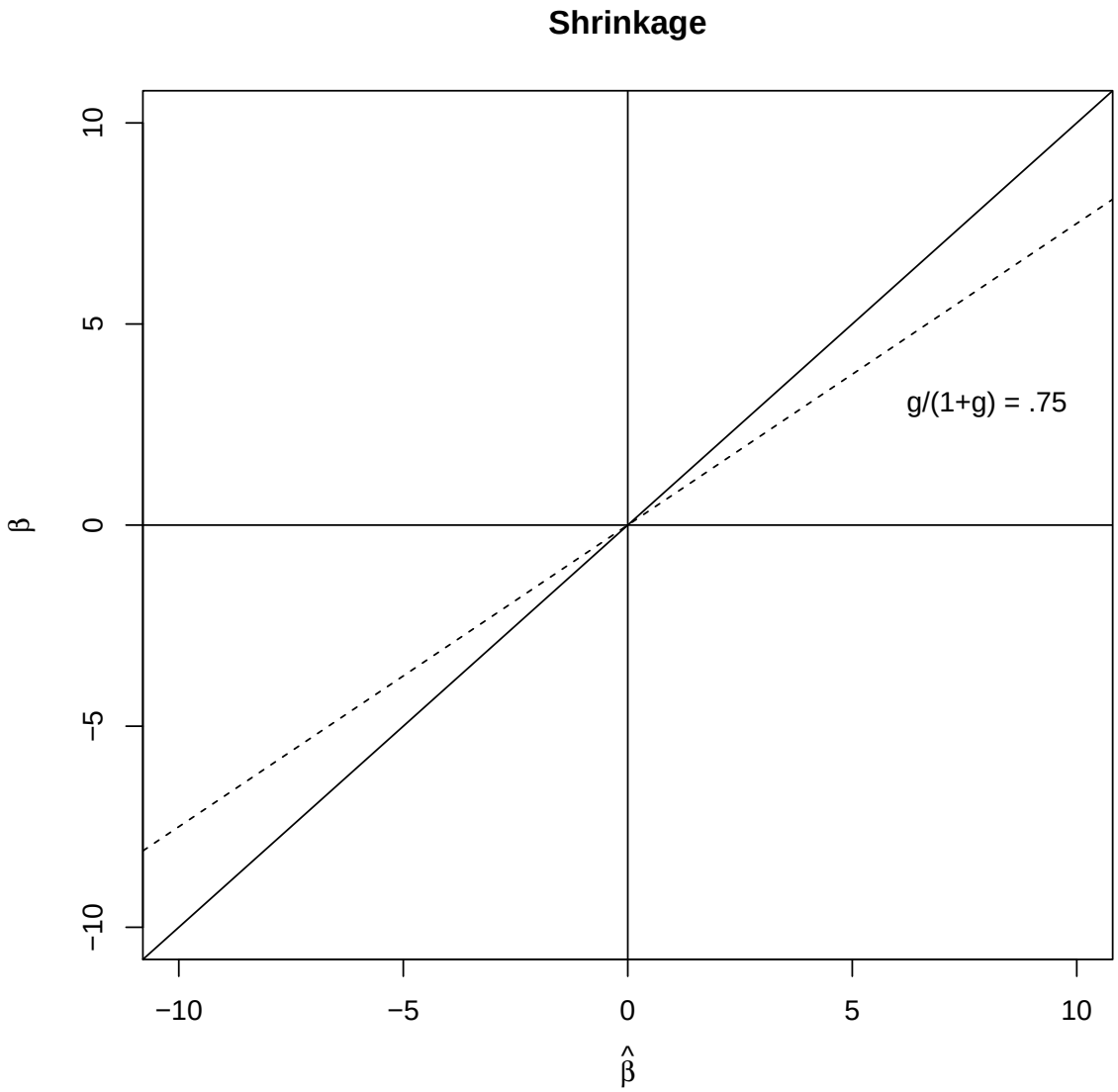
- Fixed g effect does not vanish as $n \to \infty$
- Use $g = n$ or place a prior distribution on g

Shrinkage

Posterior mean under g-prior with $\mathbf{b}_0 = 0$ $\frac{g}{1+g}\hat{\boldsymbol{\beta}}$



duke.eps

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$$\mathcal{I}(\theta) = -\mathsf{E}[\left[\frac{\partial^2 \log(\mathcal{L}(\boldsymbol{\theta}))}{\partial\theta_i \partial\theta_j}\right]]$$

Fisher Information Matrix

Log Likelihood

$$\log(\mathcal{L}(\boldsymbol{\beta}, \phi)) = \frac{n}{2}\log(\phi) - \frac{\phi}{2}\|(\mathbf{I} - \mathbf{P}_{\mathbf{x}})\mathbf{Y}\|^2 - \frac{\phi}{2}(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}})^T(\mathbf{X}^T\mathbf{X})(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}})$$

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$$\frac{\partial^2 \log \mathcal{L}}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^T} = \begin{bmatrix} -\phi(\mathbf{X}^T\mathbf{X}) & -(\mathbf{X}^T\mathbf{X})(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}) \\ -(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}})^T(\mathbf{X}^T\mathbf{X}) & -\frac{n}{2}\frac{1}{\phi^2} \end{bmatrix}$$

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$$\mathrm{E}\left[\frac{\partial^2 \log \mathcal{L}}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}^T}\right] = \begin{bmatrix} -\phi(\mathbf{X}^T\mathbf{X}) & \mathbf{0}_p \\ \mathbf{0}_p^T & -\frac{n}{2}\frac{1}{\phi^2} \end{bmatrix}$$

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\mathrm{E}[\frac{\partial^2 \log \mathcal{L}}{\partial\theta\partial\theta^T}] &= \begin{bmatrix} -\phi(\mathbf{X}^T\mathbf{X}) & \mathbf{0}_p \\ \mathbf{0}_p^T & -\frac{n}{2}\frac{1}{\phi^2} \end{bmatrix} \\
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Jeffreys Prior

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Improper prior $\int\int p_J(\boldsymbol{\beta}, \phi)\, d\boldsymbol{\beta}\, d\phi$ not finite

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Bayesian Credible Sets $p(\boldsymbol{\beta} \in C_\alpha) = 1 - \alpha$ correspond to frequentist Confidence Regions

$$\frac{\boldsymbol{\lambda}^T\boldsymbol{\beta} - \boldsymbol{\lambda}\hat{\boldsymbol{\beta}}}{\sqrt{\hat{\sigma}^2\boldsymbol{\lambda}^T(\mathbf{X}^T\mathbf{X})^{-1}\boldsymbol{\lambda}}} \sim t_{n-p}$$

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- Mixtures of Conjugate Priors