G-Priors and Mixture Distributions

STA721 Linear Models Duke University

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Bayesian Estimation

Model

$$\mathbf{Y} \sim \mathsf{N}(\mathbf{X}\boldsymbol{\beta}, \mathbf{I}_n/\phi)$$

with precision $\phi = 1/\sigma^2$.

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Default Prior Choices:

- Independent Jeffreys' Priors
- Partitioned g-priors
- Zellner-Siow Cauchy Prior, mixtures and MCMC

Readings: Hoff Chapter 9

Formal Posterior Distribution

With Independent Jeffreys Prior

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Bayesian Credible Sets $p(\boldsymbol{\beta} \in C_\alpha) = 1 - \alpha$ correspond to frequentist Confidence Regions

$$\frac{\boldsymbol{\lambda}^T\boldsymbol{\beta} - \boldsymbol{\lambda}\hat{\boldsymbol{\beta}}}{\sqrt{\hat{\sigma}^2\boldsymbol{\lambda}^T(\mathbf{X}^T\mathbf{X})^{-1}\boldsymbol{\lambda}}} \sim t_{n-p}$$

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Cannot represent anyone's prior beliefs, but used as a reference posterior

Partitioned Zellner's g-prior

Zellner recognized that some parameters might have less information

$$\mathbf{Y} = \mathbf{X}_0 \boldsymbol{\beta}_0 + \mathbf{X}_1 \boldsymbol{\beta}_1 + \boldsymbol{\epsilon}$$

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- $\boldsymbol{\beta}_1 \sim N(\mathbf{b}_1, g_1 (\mathbf{X}_1^T \mathbf{X}_1)^{-1} / \phi)$
- $p(\phi) \propto 1/\phi$

Special case $\mathbf{X}_0 = \mathbf{1}_n$ and let $g_0 \to \infty$

Bayesian Estimation with g prior

$$
\begin{aligned}
\mathbf{Y} &= \mathbf{1}\alpha_0 + \mathbf{X}_1\boldsymbol{\beta} + \boldsymbol{\epsilon} \\
p(\alpha_0, \phi) &\propto 1 \\
\boldsymbol{\beta} \mid \phi &\sim \mathsf{N}(\mathbf{0}, \frac{g}{\phi}(\mathbf{X}^T(\mathbf{I}_n - \mathbf{P_1})\mathbf{X})^{-1})
\end{aligned}
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Equivalent to

$$
\begin{aligned}
\mathbf{Y} &= \mathbf{1}\beta_0 + (I - \mathbf{P}_1)\mathbf{X}_1\boldsymbol{\beta} + \boldsymbol{\epsilon} \\
\beta_0 &= \alpha + \bar{\mathbf{x}}^T\boldsymbol{\beta} \\
p(\beta_0, \phi) &\propto 1 \\
\boldsymbol{\beta} \mid \phi &\sim \mathsf{N}(\mathbf{0}, \frac{g}{\phi}(\mathbf{X}^T(\mathbf{I}_n - \mathbf{P_1})\mathbf{X})^{-1})
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Prior Data

Note

$$(\mathbf{X}^T(\mathbf{I}_n - \mathbf{P}_1)\mathbf{X}) = (\mathbf{X}^T(\mathbf{I}_n - \mathbf{P}_1)^T(\mathbf{I}_n - \mathbf{P}_1)\mathbf{X}) = (\mathbf{X} - \mathbf{1}_n\bar{\mathbf{X}}^T)^T(\mathbf{X} - \mathbf{1}_n\bar{\mathbf{X}})$$

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Quadratic contribution to the log likelihood from prior after integrating out β_0

$$(\mathbf{Y}_c - \mathbf{X}_c\beta)^T(\mathbf{Y}_c - \mathbf{X}_c\beta) + (\beta^T \frac{\mathbf{U}^T\mathbf{U}}{g}\beta)$$

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Prior observations with $Yc = 0$.

Example: g=5, n=30

In SLR it is like an extra $Y_0 = 0$ at $\mathbf{X}_o = \sqrt{\frac{SS_x}{g}}$:

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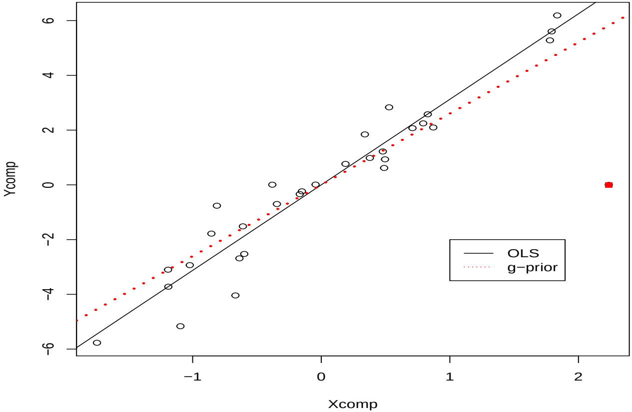

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- Mixtures of Conjugate Priors

Mixtures of Conjugate Priors

Theorem (Diaconis & Ylivisaker 1985)

Given a sampling model $p(y \mid \boldsymbol{\theta})$ *from an exponential family, any prior distribution can be expressed as a mixture of conjugate prior distributions*

- Prior $p(\boldsymbol{\theta}) = \int p(\boldsymbol{\theta} \mid \omega) p(\omega) \, d\omega$

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Zellner-Siow prior (assume $\mathbf{X}$ is centered)

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- Let $\tau = 1/g$ assign $\tau \sim G(1/2, n/2)$
- Marginal prior on $\boldsymbol{\beta} \sim C(0, \phi^{-1}\mathbf{X}^T\mathbf{X})$
- Can express posterior as a mixture of g-priors

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- What about credible intervals?

Markov Chain Monte Carlo

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$$p(\tau \mid \boldsymbol{\beta}, \phi, \mathbf{Y}) \propto \mathcal{L}(\beta_0, \boldsymbol{\beta}, \phi) \tau^{p/2} e^{(-\tau \frac{\phi}{2} \boldsymbol{\beta}^T (\mathbf{X}^T \mathbf{X}) \boldsymbol{\beta})} \tau^{1/2-1} e^{-\tau n/2}$$

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- alternate sampling from full conditional distributions given current values of other parameters. (STA 601)

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- alternate sampling from full conditional distributions given current values of other parameters. (STA 601)
- JAGS or STAN

JAGS Code: library(R2jags)

```
model = function(){
  for (i in 1:n) {
      Y[i] ~ dnorm(beta0+ (X[i] -Xbar)*beta, phi)
  }
  beta0 ~ dnorm(0, .000001*phi) #precision is 2nd arg
  beta ~ dnorm(0, phi*tau*SSX)  #precision is 2nd arg
  phi ~ dgamma(.001, .001)
  tau ~ dgamma(.5, .5*n)
  g <- 1/tau
  sigma <- pow(phi, -.5)
}
data = list(Y=Y, X=X, n =length(Y), SSX=sum(Xc^2),
            Xbar=mean(X))
ZSout = jags(data, inits=NULL,
             parameters.to.save=c("beta0","beta", "g",
                                  "sigma"),
             model=model, n.iter=10000)
```

HPD intervals

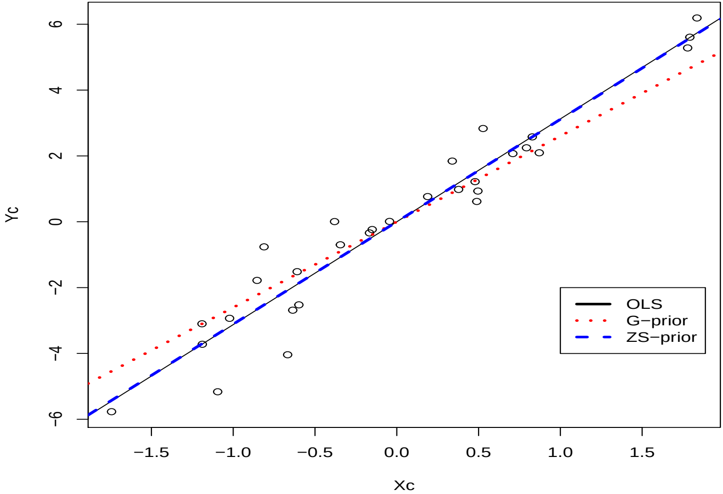
```
confint(lm(Y ~ Xc))

##                   2.5 %    97.5 %
## (Intercept) -0.3985359 0.2048303
## Xc           2.7945824 3.4555162

HPDinterval(as.mcmc(ZSout$BUGSoutput$sims.matrix))

##                lower        upper
## beta       2.7823047    3.4453690
## beta0     -0.3764027    0.2095465
## deviance  70.2043917   78.4813041
## g         19.4503373 3782.7134974
## sigma      0.6171029    1.0504892
## attr(,"Probability")
## [1] 0.95
```

Compare



```
ZSout
```

```
## Inference for Bugs model at "/var/folders/n4/nj1122xj6bn5_xgbptv7bml40000gp/T//Rtmpe7h40r/model784a5816
##  3 chains, each with 10000 iterations (first 5000 discarded), n.thin = 5
##  n.sims = 3000 iterations saved
##           mu.vect   sd.vect   2.5%     25%     50%     75%    97.5%  Rhat
## beta        3.112     0.170  2.782   2.997   3.115   3.225    3.445 1.001
## beta0      -0.099     0.152 -0.384  -0.204  -0.099   0.001    0.204 1.002
## g        2263.147 38967.029 48.273 146.129 282.298 697.063 9018.709 1.001
## sigma       0.827     0.114  0.636   0.747   0.816   0.896    1.079 1.001
## deviance   73.347     2.563 70.390  71.458  72.680  74.500   79.882 1.002
##          n.eff
## beta      3000
## beta0     1200
## g         3000
## sigma     3000
## deviance  1600
##
## For each parameter, n.eff is a crude measure of effective sample size,
## and Rhat is the potential scale reduction factor (at convergence, Rhat=1).
##
## DIC info (using the rule, pD = var(deviance)/2)
## pD = 3.3 and DIC = 76.6
## DIC is an estimate of expected predictive error (lower deviance is better).
```

Cauchy Summary

- Cauchy rejects prior mean if it is an "outlier"
- robustness related to "bounded" influence (more later)
- numerical integration or Monte Carlo sampling (MCMC)