Final Examination

Mth 230 = Sta 230[a]

Friday, December 13, 2013
9:00am – 12:00n

- This is a **closed book** exam— put your books and bags on the floor.
- You may use a calculator and **two pages** of your own notes.
- Do not share calculators or notes. **No phones** or other network-connected devices may be used as calculators or timers.
- **Show** your **work**. Neatness counts. Boxing answers helps.
- Numerical answers: **four sig**nificant **dig**its or fractions in lowest terms. For full credit you must **simplify** *all* expressions.
- Extra worksheet and pdf & normal distribution tables are attached.

Cheating on exams is a breach of trust with classmates and faculty that will not be tolerated. After completing the exam please acknowledge that you have adhered to the Duke Community Standard:

- I will not lie, cheat, or steal in my academic endeavors;
- I will conduct myself honorably in all my endeavors; and
- I will act if the Standard is compromised.

Signature: ______________________

Print Name Clearly: ______________________

1.	/20	5.	/20
2.	/20	6.	/20
3.	/20	7.	/20
4.	/20	8.	/20
	/80		/80
Total:	/160		

Problem 1: Cuddles the kitten is equally likely to climb up onto the roof (and get stuck there) or to climb too high in the front-yard oak tree (and get stuck there), but the time (in hours) T it takes before he finds the courage to climb down again has a different distribution for these two cases:

$$\begin{aligned} \mathsf{P}[T > t \mid \text{ Roof }] &= e^{-t/10}, & t > 0 \\ \mathsf{P}[T > t \mid \text{ Tree }] &= 10/(10+t), & t > 0. \end{aligned}$$

a) (5) Find the probability density function for T (reflecting both possibilities), correctly for all times $-\infty < t < \infty$:

$f(t) =$

b) (5) Given that Cuddles has been missing for $t = 5$ hours,[1] what is the probability he's up the tree?

$\mathsf{P}[\text{ Tree } \mid T > 5] =$

c) (5) If Cuddles returns after exactly $t = 20$ hours, what is the probability he spent those hours on the roof?

$\mathsf{P}[\text{ Roof } \mid T = 20] =$

d) (5) Find the expected length (in hours) of Cuddles' adventure:

$\mathsf{E}[T] =$

[1] You may assume he's missing because he climbed the tree or climbed onto the roof.

Problem 2: With the same Cuddles as in Problem 1, choose the best probability distribution for each random variable below from among the choices *Binomial, Exponential, Gamma, Geometric, Hypergeometric, Negative Binomial, Normal, Poisson, Uniform, or Weibull* and, whatever the distribution, give its mean μ (specify the units!):

a) (4) The number of times Cuddles climbs the tree before he climbs onto the roof, if his choices are equally-likely and independent:
◯ Bi ◯ Ex ◯ Ga ◯ Ge ◯ HG ◯ NB ◯ No ◯ Po ◯ Un ◯ We
$\mu =$

b) (4) The number of mice he catches in one week, if he catches about one every 12 hours and if catches in disjoint time intervals are independent:
◯ Bi ◯ Ex ◯ Ga ◯ Ge ◯ HG ◯ NB ◯ No ◯ Po ◯ Un ◯ We
$\mu =$

c) (4) The length of time (choose and specify the units) until he catches two mice, with the same assumptions as above:
◯ Bi ◯ Ex ◯ Ga ◯ Ge ◯ HG ◯ NB ◯ No ◯ Po ◯ Un ◯ We
$\mu =$

d) (4) The total weight (in *lb*) of mice he catches in a year, with the same assumptions as above, if the average mouse weighs 2 *oz*? (1 *lb* = 16 *oz*)
◯ Bi ◯ Ex ◯ Ga ◯ Ge ◯ HG ◯ NB ◯ No ◯ Po ◯ Un ◯ We
$\mu =$

e) (4) The number of times he climbs the tree before his second time climbing the roof, with the same assumptions as above?
◯ Bi ◯ Ex ◯ Ga ◯ Ge ◯ HG ◯ NB ◯ No ◯ Po ◯ Un ◯ We
$\mu =$

Problem 3: Three events A, B, C have probabilities a, b, and c respectively. For all parts of this problem the events A and B are independent; the events B and C are disjoint.

a) (4) What is the probability of $(A \cap B) \cup C$?

b) (4) If A and C are also independent, find $\mathsf{P}[A \cup B \cup C]$.

c) (4) If A and C are disjoint, find $\mathsf{P}[A \cup B \cup C]$.

d) (4) If A and C are disjoint, find $\mathsf{P}[A \mid B \cup C]$.

e) (4) If $a = 1/3$ and $b = c = 1/2$, what is $\mathsf{P}[A \cap C]$?

Problem 4:

The independent random variables X and Y are Poisson-distributed with means $\mathsf{E}X = 2$ and $\mathsf{E}Y = 3$, respectively, so their sum $Z = X + Y$ also has a Poisson distribution.

a) (6) Let $g(n) = (-1)^n$ for integers $n \ge 0$. Find:

$\mathsf{E}\,[g(X)] =$ ____________________ $\mathsf{P}[X \text{ is even }] =$ ____________________

b) (5) Find the indicated conditional probability, for each integer $x \in \mathbb{Z}$:

$\mathsf{P}[X = x \mid Z = 10] =$ ____________________

c) (5) Find the indicated conditional probability:

$\mathsf{P}[X \le 2 \mid X > 0] =$ ____________________

d) (4) Find:

$\mathsf{E}[XYZ] =$ ____________________

Problem 5: When Cuddles the kitten attacks a ball of string it rolls down the hall unwinding a length X (in meters) of string, which we will model with the exponential distribution, $X \sim \mathsf{Ex}(0.2)$.

a) (5) Find:

$\mathsf{P}[X \leq 10\,\mathrm{m}] =$ ______________

b) (5) Find the pdf for $Y := X^2$, correctly for all $-\infty < y < \infty$:

$f_Y(y) =$ ______________

c) (5) Find the pdf for $Z := \sqrt{X}$ for all $z \in \mathbb{R}$:

$f_Z(z) =$ ______________

d) (5) The area A (in m^2) of the largest circle you can make with the unwound length of string has one of the distributions listed on page 13 at the end of this exam. Which one, and with what parameter(s)?

$A \sim$ ______________

Problem 6: The random variables X and Y have joint probability density function

$$f(x,y) = \frac{1}{3} x y e^{-y}\ \mathbf{1}_{\{0<x<y<\infty\}}$$

a) (6) Find the marginal density functions:

$f_X(x) =$ $f_Y(y) =$

b) (4) Find the conditional pdf:

$f_{X|Y}(x \mid y) =$

Problem 6 (cont):

Recall $X, Y \sim f(x,y) = \frac{1}{3} x y e^{-y} \mathbf{1}_{\{0<x<y<\infty\}}$.

c) (4) Find the other conditional pdf:

$f_{Y|X}(y \mid x) =$

d) (6) Find:

$\mathsf{P}[Y > 2X] =$ ____________________

Problem 7: $X \sim \mathsf{No}(\mu = 2, \sigma^2 = 9)$ and $Y \sim \mathsf{No}(\mu = 5, \sigma^2 = 16)$ have normal distributions, with correlation coefficient $\rho = 3/8$. Their sum and difference are $S := X + Y$ and $T := Y - X$.

a) (5) What are the mean and variance of S?

$\mathsf{E}[S] =$ ______________ $\mathsf{V}[S] =$ ______________

b) (5) Find the indicated probability:

$\mathsf{P}[X > Y] =$ ______________

c) (5) Find the covariance:

$\mathsf{Cov}(S, T) =$ ______________

d) (5) Find the indicated conditional probability[2]:

$\mathsf{P}[X \le 0 \mid Y = 1] =$ ______________

[2] We're not really conditioning on the *event* $[Y = 1]$, which has probability zero; rather, we're finding the probability that $X \le 0$ under the conditional *distribution* $f_{X|Y}(x \mid y)$.

Problem 8: True or false? Circle one; each answer is 2 points. No explanations are needed, but you can give one if the question seems ambiguous to you.

a) T F If A and B are independent, then so are A^c and B^c.

b) T F If $\{X_j\}$ are iid with mean μ and variance σ^2 then $\bar{X}_n := \frac{1}{n}\sum_{j=1}^n X_j$ is approximately normally distributed with mean μ and variance σ^2/n, for large n.

c) T F If $X \sim \mathsf{Ge}(p)$ has a geometric distribution with mean $(1-p)/p$, then $\mathsf{P}[X \text{ is even }] = 1/(2-p)$.

d) T F If a product lifetime T (in months) has distribution given by $\mathsf{P}[T > t] = e^{-t^2/100}$ for $t > 0$, then T^2 has an exponential distribution.

e) T F If a product lifetime T (in months) has distribution given by $\mathsf{P}[T > t] = e^{-t^2/100}$ for $t > 0$, then a used unit has a better chance of surviving one more year than a new one.

f) T F If $\mathsf{Cov}(X,Y) = 0$ then $\mathsf{E}[X \cdot Y] = \mathsf{E}[X] \cdot \mathsf{E}[Y]$

g) T F If $\{X_i\} \overset{\text{iid}}{\sim} \mathsf{Ge}(p)$ have independent Geometric distributions then $\sum_{i=1}^n X_i$ has a Hypergeometric distribution.

h) T F If $X > 0$ has a continuous distribution with pdf $f_X(x)$, then $Y \equiv X^2$ has pdf $f_Y(y) = f_X(\sqrt{y})/(2\sqrt{y})$.

i) T F If $\mathsf{E}[X \cdot Y] = \mathsf{E}[X] \cdot \mathsf{E}[Y]$ then X and Y are independent.

j) T F If $\{X_j\}$ are iid with mean μ and variance σ^2 then $S_n := \sum_{j=1}^n X_j$ has mean $n\mu$ and variance $n^2\sigma^2$.

Extra worksheet, if needed:

Another extra worksheet, if needed:

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz:$$

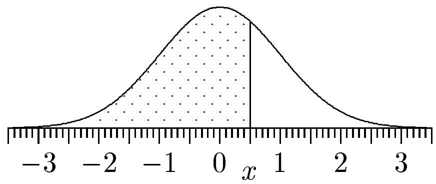

Table 5.1 Area $\Phi(x)$ under the Standard Normal Curve to the left of x.

x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

$\Phi(0.6745) = 0.75$ $\Phi(1.6449) = 0.95$ $\Phi(2.3263) = 0.99$ $\Phi(3.0902) = 0.999$

$\Phi(1.2816) = 0.90$ $\Phi(1.9600) = 0.975$ $\Phi(2.5758) = 0.995$ $\Phi(3.2905) = 0.9995$

Name	Notation	pdf/pmf	Range	Mean μ	Variance σ^2	
Beta	$\mathsf{Be}(\alpha,\beta)$	$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$x\in(0,1)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	
Binomial	$\mathsf{Bi}(n,p)$	$f(x)=\binom{n}{x}p^x q^{(n-x)}$	$x\in 0,\cdots,n$	$n\,p$	$n\,p\,q$	$(q=1-p)$
Exponential	$\mathsf{Ex}(\lambda)$	$f(x)=\lambda\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	$1/\lambda$	$1/\lambda^2$	
Gamma	$\mathsf{Ga}(\alpha,\lambda)$	$f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	α/λ	α/λ^2	
Geometric	$\mathsf{Ge}(p)$	$f(x)=p\,q^x$	$x\in\mathbb{Z}_+$	q/p	q/p^2	$(q=1-p)$
		$f(y)=p\,q^{y-1}$	$y\in\{1,...\}$	$1/p$	q/p^2	$(y=x+1)$
HyperGeo.	$\mathsf{HG}(n,A,B)$	$f(x)=\frac{\binom{A}{x}\binom{B}{n-x}}{\binom{A+B}{n}}$	$x\in 0,\cdots,n$	$n\,P$	$n\,P\,(1-P)\frac{N-n}{N-1}$	$(P=\frac{A}{A+B})$
Logistic	$\mathsf{Lo}(\mu,\beta)$	$f(x)=\frac{e^{-(x-\mu)/\beta}}{\beta[1+e^{-(x-\mu)/\beta}]^2}$	$x\in\mathbb{R}$	μ	$\pi^2\beta^2/3$	
Log Normal	$\mathsf{LN}(\mu,\sigma^2)$	$f(x)=\frac{1}{x\sqrt{2\pi\sigma^2}}e^{-(\log x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}_+$	$e^{\mu+\sigma^2/2}$	$e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right)$	
Neg. Binom.	$\mathsf{NB}(\alpha,p)$	$f(x)=\binom{x+\alpha-1}{x}p^\alpha\,q^x$	$x\in\mathbb{Z}_+$	$\alpha q/p$	$\alpha q/p^2$	$(q=1-p)$
		$f(y)=\binom{y-1}{y-\alpha}p^\alpha\,q^{y-\alpha}$	$y\in\{\alpha,...\}$	α/p	$\alpha q/p^2$	$(y=x+\alpha)$
Normal	$\mathsf{No}(\mu,\sigma^2)$	$f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}$	μ	σ^2	
Pareto	$\mathsf{Pa}(\alpha,\epsilon)$	$f(x)=\alpha\,\epsilon^\alpha/x^{\alpha+1}$	$x\in(\epsilon,\infty)$	$\frac{\epsilon\,\alpha}{\alpha-1}$	$\frac{\epsilon^2\alpha}{(\alpha-1)^2(\alpha-2)}$	
Poisson	$\mathsf{Po}(\lambda)$	$f(x)=\frac{\lambda^x}{x!}e^{-\lambda}$	$x\in\mathbb{Z}_+$	λ	λ	
Snedecor F	$F(\nu_1,\nu_2)$	$f(x)=\frac{\Gamma(\frac{\nu_1+\nu_2}{2})(\nu_1/\nu_2)^{\nu_1/2}}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})}\times$ $x^{\frac{\nu_1-2}{2}}\left[1+\frac{\nu_1}{\nu_2}x\right]^{-\frac{\nu_1+\nu_2}{2}}$	$x\in\mathbb{R}_+$	$\frac{\nu_2}{\nu_2-2}$	$\left(\frac{\nu_2}{\nu_2-2}\right)^2\frac{2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)}$	
Student t	$t(\nu)$	$f(x)=\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}}[1+x^2/\nu]^{-(\nu+1)/2}$	$x\in\mathbb{R}$	0	$\nu/(\nu-2)$	
Uniform	$\mathsf{Un}(a,b)$	$f(x)=\frac{1}{b-a}$	$x\in(a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	
Weibull	$\mathsf{We}(\alpha,\beta)$	$f(x)=\alpha\beta\,x^{\alpha-1}\,e^{-\beta\,x^\alpha}$	$x\in\mathbb{R}_+$	$\frac{\Gamma(1+\alpha^{-1})}{\beta^{1/\alpha}}$	$\frac{\Gamma(1+2/\alpha)-\Gamma^2(1+1/\alpha)}{\beta^{2/\alpha}}$	