Statistics 360/601 – Modern Bayesian Theory

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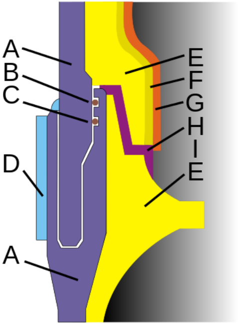
Lecture 20something - Nov 27, 2018

The Challenger disaster

- This is an example from Monte Carlo Statistical Methods by Robert and Casella (2004).

- In 1986, the Challenger space shuttle exploded on take off.
- Suspected cause of explosion: O-ring failure.



By Kapitel, vectorisation by Adam Rdzikowski - Own work, GFDL, curid=24312838

Simplified cross section of the joints between rocket segments SRB; outside to left.
Legend:

- A - steel wall 0.5 inches (12.7 mm) thick
- B - base O-ring gasket,
- C - backup O-ring gasket,
- D - Strengthening-Cover band,
- E - insulation,
- F - insulation,
- G - carpeting,
- H - sealing paste,
- I - fixed propellant

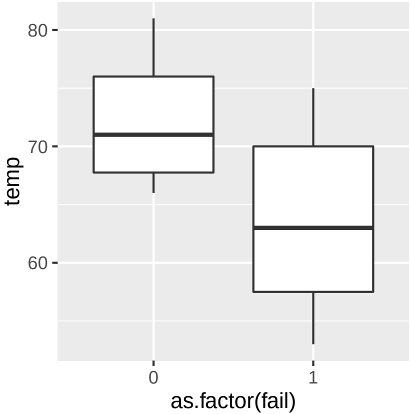
The Challenger disaster

- O-rings are rigorously tested.
- Anomaly during the launch: unusually cold temperature (31 degrees F).
- There is reason to believe that O-ring failure probabilities increase as temperature decreases.
- We have data on 23 other launches:

Flight	14	9	23	10	1	5	13	15	4	3	8	17	2	11	6	7	16	21	19	22	12	20	18
Failure	1	1	1	1	0	0	0	0	0	0	0	0	1	1	0	0	0	1	0	0	0	0	0
Temperature	53	57	58	63	66	67	67	67	68	69	70	70	70	70	72	73	75	75	76	76	78	79	81

- Hard to read...
- Looks like more 1s at lower temperatures.

The Challenger disaster



Linear relationship between odds or log odds of failure and temperature seems reasonable. Use logistic regression!

The Challenger disaster

Our model:

$$\text{Pr}(\text{failure} = 1) = \frac{\exp(\alpha + \beta \times \text{temp})}{1 + \exp(\alpha + \beta \times \text{temp})}$$

We have 23 data points that we will assume are conditionally independent.

$$\text{failure}_1, \ldots, \text{failure}_{23} | \alpha, \beta, \text{temp}_1, \ldots, \text{temp}_{23} \sim$$
$$\prod \left(\frac{\exp(\alpha + \beta \times \text{temp}_i)}{1 + \exp(\alpha + \beta \times \text{temp}_i)}\right)^{\text{failure}_i} \left(\frac{1}{1 + \exp(\alpha + \beta \times \text{temp}_i)}\right)^{1-\text{failure}_i}$$

Metropolis-Hastings

Dissecting the code

- Initialize $\theta^{(0)}$
- Propose $\theta^{\star}$ from proposal $J_{\theta}(\theta|\theta^{(0)})$
- Evaluate the likelihood under the proposal $v_1 = \prod p(y_i|\theta^{\star})$ and under the current $\theta^{(0)}$: $v_2 = \prod p(y_i|\theta^{(0)})$
- Evaluate the prior under the proposal $v_3 = \pi_{\theta}(\theta^{\star})$ and under the current value $v_4 = \pi_{\theta}(\theta^{(0)})$
- Evaluate the proposal probability for the proposed value $v_5 = J_{\theta}(\theta^{\star}|\theta^{(0)})$ and for the current value $v_6 = J_{\theta}(\theta^{(0)}|\theta^{\star})$
- Compute the acceptance ratio: $r = \min(1, \frac{v_1}{v_2}\frac{v_3}{v_4}\frac{v_6}{v_5})$
- Simulate $u \sim unif(0,1)$ and accept $\theta^{\star}$ if $u < r$.

Metropolis-Hastings

Dissecting the code

- Initialize $\theta^{(0)} = (\alpha^{(0)}, \beta^{(0)})$
- Propose $\theta^\star$ from proposal $\pi_\alpha(\alpha|\hat{b})\phi(\beta)$.
- Evaluate the likelihood under the proposal: $v_1 = \prod p(y_i|\alpha^\star, \beta^\star, x_i)$ and under $\theta^{(0)}$: $v_2 = \prod p(y_i|\alpha^{(0)}, \beta^{(0)}, x_i)$
- Evaluate the prior under the proposal $v_3 = \pi_\alpha(\alpha^\star|\hat{b})\pi_\beta(\beta^\star)$ and under the current value $v_4 = \pi_\alpha(\alpha^{(0)}|\hat{b})\pi_\beta(\beta^{(0)})$
- Evaluate the proposal probability for the proposed value $v_5 = \pi_\alpha(\alpha^\star|\hat{b})\phi(\beta^\star)$ and for the current value $v_6 = \pi_\alpha(\alpha^{(0)}|\hat{b})\phi(\beta^{(0)})$
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- Propose $\theta^{\star}$ from proposal $\pi_{\alpha}(\alpha|\hat{b})\phi(\beta)$.
- Evaluate the likelihood under the proposal:
  $v_1 = \prod p(y_i|\alpha^{\star}, \beta^{\star}, x_i)$ and under $\theta^{(0)}$:
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- Compute the acceptance ratio: $r = \min(1, \frac{v_1}{v_2}\cancel{\frac{v_3}{v_4}}\cancel{\frac{v_6}{v_5}}\frac{\phi(\beta^{(0)})}{\phi(\beta^{\star})})$
- Simulate $u \sim unif(0, 1)$ and accept $\theta^{\star}$ if $u < r$.

Metropolis-Hastings

Proposal and evaluation

```
> rprop <- function(theta0) {
+   a <- log(rexp(1, 1 / exp(a.mle + 0.577)))
+   b <- rnorm(1, b.mle, sqrt(0.1^2))
+   return(c(a, b))
+ }
>
> dprop <- function(theta, theta0) {
+   a <- theta[1]
+   b <- theta[2]
+   pr1 <- exp(a) * exp(-exp(a) / b.mme) / b.mme
+   pr2 <- dnorm(b, b.mle, sqrt(var.b.mle))
+   return(pr1 * pr2)
+ }
```

Metropolis-Hastings

Posterior evaluation

```
> dpost <- function(theta) {
+   a <- theta[1]
+   b <- theta[2]
+   p <- 1 - 1 / (1 + exp(a + b * x))
+   lik <- exp(sum(dbinom(y, size=1, prob=p, log=TRUE)))
+   dprior <- exp(a) * exp(-exp(a) / b.mme) / b.mme
+   return(lik * dprior)
+ }
```

Metropolis-Hastings

One iteration

```
> theta_star <- rprop(theta[i-1,])
> r <- log(dpost(theta_star)) -
+ log(dpost(theta[i-1,])) +
+ log(dprop(theta[i-1,],theta_star)) -
+ log(dprop(theta_star,theta[i-1,]))
> r <- min(exp(r),1)
> if(runif(1)<= R){
+   theta[i,] <- theta_star; ct <- ct + 1}
> else{ theta[i,] <- theta[i-1,] }
```