Lecture 20something – Rank likelihood

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Outline

Ordered outcomes in Bayesian infernece

Rank likelihood

Likelihoods for Fixed Rank Nomination Networks with Applications to Friendship Networks from Add Health

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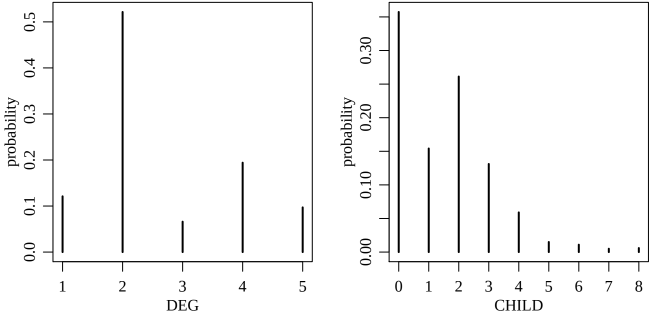
- Coding $1 =$ "high school," 2 "college," and so on...
- College is not twice as much as high school.
- We call these type of outcomes "ordinal non-numeric".

Ordered outcomes

Data on education level and number of children from the 1994 General Social Survey.



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Chapter 12.1.1 in Hoff

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- If we know how to get from "education level" to "education" then we can just model that in a numeric way.
- If we can't go directly we can treat the "education" variable as latent:

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- What can be in β?

  The location can be defined by g.

Probit regression

Chapter 12.1.1 in Hoff

When the observed variable has K categories, g can be defined as:

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y = g(z) &= 1 \text{ if } \quad -\infty = g_0 < z < g_1 \\
&= 2 \text{ if } \quad g_1 < z < g_2 \\
&\vdots \\
&= K \text{ if } \quad g_{K-1} < z < g_K \quad = \infty
\end{aligned}
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- Bayesian approach: normal priors.
- Get joint posterior via a Gibbs sampler.

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Full model

Model:

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Priors:

$$\beta \sim \text{normal}(0, n(X^t X)^{-1})$$
$$\{g_1, \ldots, g_{K-1}\} \sim p(g)(= \text{normal}(\mu, \Sigma))$$

Probit regression

Full conditionals: β

- Just like with ordinary regression: $p(\beta|y,z,g) \propto p(\beta)p(z|\beta)$.
- No dependence on g.
- Normal-normal conjugacy, so posterior is also normal:

$$\mathrm{var}(\beta|z) = \frac{n}{n+1}(X^tX)^{-1}$$
$$\mathrm{E}(\beta|z) = \frac{n}{n+1}(X^tX)^{-1}X^tz.$$

Probit regression

Full conditionals: Z

- The sampling distribution for the Z_i is $\mathrm{normal}(\beta^t x_i, 1)$.
- The posterior of the Z_i is simply that constrained to be in the "correct" subinterval.
- If $Y_i = y_i$ then Z_i must be in the interval $(g_{y_i - 1}, g_{y_i})$.
- The posterior is a constrained normal.

Probit regression

Full conditionals: g

- Recall the prior on g is multivariate normal with mean vector μ and diagonal covariance matrix Σ.
- Conditional on all the Y and Z, the element g_k must lie between all the z_i for which $y_i = k$ and the z_i for which $y_i = k + 1$.
- With a normal prior, this is another constrained normal.

Sampling from a constrained normal

- We are interested in sampling g from a normal distribution with mean μ, variance σ constrained to the interval $(a, b) \in \bar{\mathbb{R}}$.
- Easiest way to do this by sampling a uniform random variable and doing an inverse CDF transform:

$$u \sim \mathrm{Unif}(\Phi(\frac{a-\mu}{\sigma}), \Phi(\frac{b-\mu}{\sigma}))$$
$$g = \mu + \sigma\Phi^{-1}(u)$$

- R code is given by:

```
u <- runif(1, pnorm((a-mu)/sigma),pnorm((b-mu)/sigma))
g = mu + sigma*qnorm(u)
```

Example

GSS data

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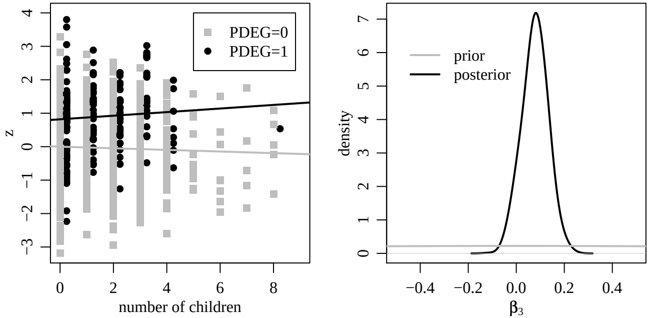

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- For discrete data, there is less information due to the possibility of ties.

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- Theoretical guarantees (Bickel and Ritov, 1997, Hoff, Niu and Wellner, 2014).

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- Posterior inference for Z_i does change since we must update each of the Z_i with respect to the other ones (there are no more cutoffs).
- Posterior of Z_i conditional on the ordering and the value of the other Z_j and on β is a constrained normal with boundary given by $\max\{z_j : y_j < y_i\}$ and $\min\{z_j : y_i < y_j\}$.

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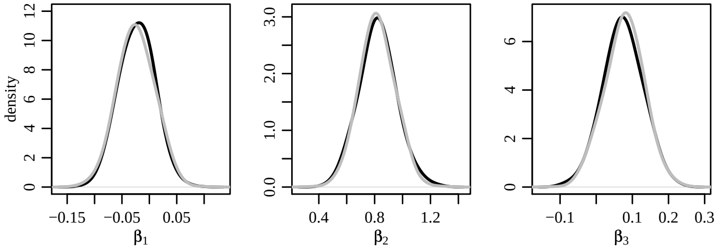

Rank likelihood

Applications

- Semiparametric regression.
- Ordinal regression.
- Semiparameteric copula estimation (Section 12.2).
- Network analysis.

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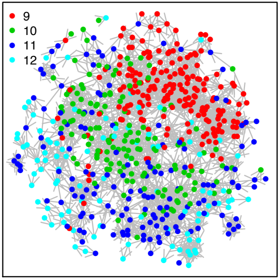

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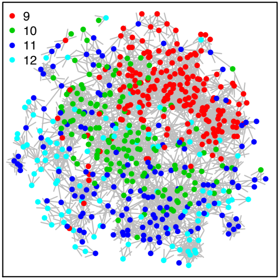

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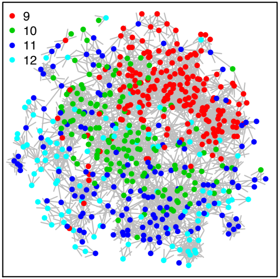

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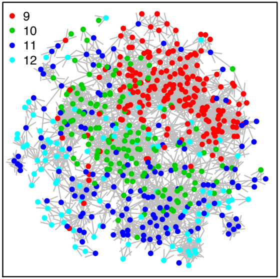

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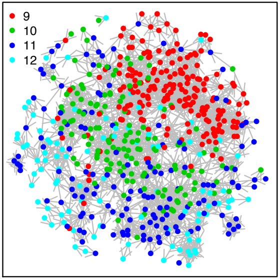

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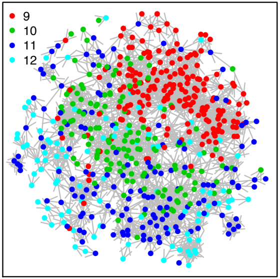

A type of likelihood that accommodates the ranked and censored nature of data from Fixed Rank Nomination (FRN) surveys and allows for estimation of regression effects.

FRN Outline

- Fixed rank nominations likelihood
- Rank based likelihood
- Binary likelihood
- Simulations
- AddHealth example

Notation

- $Y = \{y_{ij} : i \neq j\}$ is a sociomatrix of ordinal relationships

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  Observed outdegree $d_i = \sum_{j \neq i} 1\,(s_{ij} > 0)$ satisfied $d_i \leq m$

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  Observed outdegree $d_i = \sum_{j \neq i} 1\left(s_{ij} > 0\right)$ satisfied $d_i \leq m$
- For each likelihood, define the set relations between s_{ij} and y_{ij}

Notation

- $Y = \{y_{ij} : i \neq j\}$ is a sociomatrix of ordinal relationships

  $y_{ij} > y_{ik}$ denotes person i preferring person j to person k

$$Y = \begin{pmatrix} - & y_{12} & \cdots & y_{1n} \\ y_{21} & - & & \\ \vdots & & - & \\ y_{n1} & & & - \end{pmatrix}$$

- In FRN we observe a sociomatrix $S = \{s_{ij} : i \neq j\}$

  $s_{ij} = 0$ if j is not nominated by i

  $s_{ij} > s_{ik}$ if i scores j more highly than k

  Observed outdegree $d_i = \sum_{j \neq i} 1\left(s_{ij} > 0\right)$ satisfied $d_i \leq m$
- For each likelihood, define the set relations between s_{ij} and y_{ij}
- Statistical model $\{p\left(Y|\theta\right) : \theta \in \Theta\}$ assists in analysis

Model - standard approach (SRM)

- The social relations model (SRM) was introduced by Warner, Kenny and Stoto (1979).
- Probit model with row and column effects:

$$\begin{aligned} Y_{ij} &= \mathbf{1}_{Z_{ij}>0} \\ Z_{ij} &= \beta^t X_{ij} + a_i + b_j + \epsilon_{ij} \\ (a_i\ b_i) &\overset{\text{iid}}{\sim} \text{normal}\left(0, \Sigma_{ab}\right) \\ \text{cor}\left(\epsilon_{ij}, \epsilon_{ji}\right) &= \rho \end{aligned}$$

- Note that $\text{cov}\left(Z_{ij}, Z_{kl}\right) = 0$ unless Z_{ij} and Z_{kl} are in the same column, same row, or are reciprocal.

Model - extended approach

Multiplicative effects

$$
\begin{aligned}
Y_{ij} &= \mathbf{1}_{Z_{ij}>0} \\
Z_{ij} &= \beta^t X_{ij} + a_i + b_j + \mathbf{u_i^t v_j} \epsilon_{ij} \\
(a_i\ b_i) &\overset{\text{iid}}{\sim} \text{normal}\,(0, \Sigma_{ab}) \\
(u_i, v_i) &\sim \text{normal}(0, \Sigma_{uv}) \\
\text{cor}\,(\epsilon_{ij}, \epsilon_{ji}) &= \rho
\end{aligned}
$$

Set relations

$$\begin{aligned}
s_{ij} > s_{ik} &\quad \Rightarrow y_{ij} > y_{ik} \\
s_{ij} = 0 \text{ and } d_i < m &\quad \Rightarrow y_{ij} \leq 0 \\
s_{ij} > 0 &\quad \Rightarrow y_{ij} > 0 \\
s_{ij} = 0 &\quad \Rightarrow y_{ij} < 0
\end{aligned}$$

Set relations

$$
\begin{aligned}
s_{ij} > s_{ik} &\quad \Rightarrow y_{ij} > y_{ik} \\
s_{ij} = 0 \text{ and } d_i < m &\quad \Rightarrow y_{ij} \leq 0 \\
s_{ij} > 0 &\quad \Rightarrow y_{ij} > 0 \\
s_{ij} = 0 &\quad \Rightarrow y_{ij} < 0
\end{aligned}
$$

s_i	___	___	___	___	___	___	___	___	___	___
y_i	___	___	___	___	___	___	___	___	___	___
	1	2	3	4	5	6	7	8	9	10

Set relations

$$\begin{aligned} s_{ij} > s_{ik} &\Rightarrow y_{ij} > y_{ik} \\ s_{ij} = 0 \text{ and } d_i < m &\Rightarrow y_{ij} \leq 0 \\ s_{ij} > 0 &\Rightarrow y_{ij} > 0 \\ s_{ij} = 0 &\Rightarrow y_{ij} < 0 \end{aligned}$$

s_i	4	3	2	1	0	0	0	0	0	0
y_i										
	1	2	3	4	5	6	7	8	9	10

Set relations

$$
\begin{aligned}
s_{ij} > s_{ik} &\Rightarrow y_{ij} > y_{ik} \\
s_{ij} = 0 \text{ and } d_i < m &\Rightarrow y_{ij} \leq 0 \\
s_{ij} > 0 &\Rightarrow y_{ij} > 0 \\
s_{ij} = 0 &\Rightarrow y_{ij} < 0
\end{aligned}
$$

s_i	4	3	2	1	0	0	0	0	0	0
y_i	y_{i1}	y_{i2}	y_{i3}	y_{i4}	$0>$	$0>$	$0>$	$0>$	$0>$	$0>$
	1	2	3	4	5	6	7	8	9	10

Set relations

$$
\begin{aligned}
s_{ij} > s_{ik} &\quad \Rightarrow y_{ij} > y_{ik} \\
s_{ij} = 0 \text{ and } d_i < m &\quad \Rightarrow y_{ij} \leq 0 \\
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$$

s_i										
y_i										
	1	2	3	4	5	6	7	8	9	10

Set relations

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s_i	5	4	3	2	1	0	0	0	0	0
y_i										
	1	2	3	4	5	6	7	8	9	10

Set relations

$$\begin{aligned} s_{ij} > s_{ik} &\Rightarrow y_{ij} > y_{ik} \\ s_{ij} = 0 \text{ and } d_i < m &\Rightarrow y_{ij} \leq 0 \\ s_{ij} > 0 &\Rightarrow y_{ij} > 0 \\ s_{ij} = 0 &\Rightarrow y_{ij} < 0 \end{aligned}$$

s_i	5	4	3	2	1	0	0	0	0	0
y_i	y_{i1}	y_{i2}	y_{i3}	y_{i4}	y_{i5}	?	?	?	?	?
	1	2	3	4	5	6	7	8	9	10

Set relations

$$\begin{aligned} s_{ij} > s_{ik} &\quad \Rightarrow y_{ij} > y_{ik} \\ s_{ij} = 0 \text{ and } d_i < m &\quad \Rightarrow y_{ij} \leq 0 \\ s_{ij} > 0 &\quad \Rightarrow y_{ij} > 0 \\ s_{ij} = 0 &\quad \Rightarrow y_{ij} < 0 \end{aligned}$$

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\end{aligned}
$$

- Define sets $F(S)$, $R(S)$ and $B(S)$

Set relations

$$
\begin{aligned}
s_{ij} > s_{ik} \quad &\Rightarrow y_{ij} > y_{ik} \\
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$$

- Define sets $F(S)$, $R(S)$ and $B(S)$
- Define a model $\{p(Y|\theta) : \theta \in \Theta\}$

Set relations

$$\begin{aligned} s_{ij} > s_{ik} &\Rightarrow y_{ij} > y_{ik} \\ s_{ij} = 0 \text{ and } d_i < m &\Rightarrow y_{ij} \leq 0 \\ s_{ij} > 0 &\Rightarrow y_{ij} > 0 \\ s_{ij} = 0 &\Rightarrow y_{ij} < 0 \end{aligned}$$

- Define sets $F(S)$, $R(S)$ and $B(S)$
- Define a model $\{p(Y|\theta) : \theta \in \Theta\}$
- Base inference on θ on a likelihood $\int dP(Y|\theta)$

FRN

$$\left.\begin{array}{rl} s_{ij} > s_{ik} & \Rightarrow y_{ij} > y_{ik} \\ s_{ij} = 0 \text{ and } d_i < n & \Rightarrow y_{ij} \leq 0 \\ s_{ij} > 0 & \Rightarrow y_{ij} > 0 \end{array}\right\} F(S)$$

$$s_{ij} = 0 \quad \Rightarrow y_{ij} < 0$$

FRN

$$\left.\begin{array}{rl} s_{ij} > s_{ik} & \Rightarrow y_{ij} > y_{ik} \\ s_{ij} = 0 \text{ and } d_i < n & \Rightarrow y_{ij} \leq 0 \\ s_{ij} > 0 & \Rightarrow y_{ij} > 0 \end{array}\right\} F(S)$$

$$s_{ij} = 0 \quad \Rightarrow y_{ij} < 0$$

- Captures censoring in the data

FRN

$$\left.\begin{aligned} s_{ij} > s_{ik} &\Rightarrow y_{ij} > y_{ik} \\ s_{ij} = 0 \text{ and } d_i < n &\Rightarrow y_{ij} \leq 0 \\ s_{ij} > 0 &\Rightarrow y_{ij} > 0 \end{aligned}\right\} F(S)$$

$$s_{ij} = 0 \Rightarrow y_{ij} < 0$$

- Captures censoring in the data
- Differentiates between different ranks

Rank

$$\begin{aligned} s_{ij} > s_{ik} &\Rightarrow y_{ij} > y_{ik} \, \} \; R(S) \\ s_{ij} = 0 \text{ and } d_i < n &\Rightarrow y_{ij} \leq 0 \\ s_{ij} > 0 &\Rightarrow y_{ij} > 0 \\ s_{ij} = 0 &\Rightarrow y_{ij} < 0 \end{aligned}$$

Rank

$$\begin{aligned} s_{ij} > s_{ik} &\Rightarrow y_{ij} > y_{ik} \} \; R(S) \\ s_{ij} = 0 \text{ and } d_i < n &\Rightarrow y_{ij} \leq 0 \\ s_{ij} > 0 &\Rightarrow y_{ij} > 0 \\ s_{ij} = 0 &\Rightarrow y_{ij} < 0 \end{aligned}$$

- Valid but not fully informative: $F(S) \subsetneq R(S)$

Rank

$$\begin{aligned} s_{ij} > s_{ik} \quad & \Rightarrow y_{ij} > y_{ik} \} \; R(S) \\ s_{ij} = 0 \text{ and } d_i < n \quad & \Rightarrow y_{ij} \leq 0 \\ s_{ij} > 0 \quad & \Rightarrow y_{ij} > 0 \\ s_{ij} = 0 \quad & \Rightarrow y_{ij} < 0 \end{aligned}$$

- Valid but not fully informative: $F(S) \subsetneq R(S)$
- Variants of this likelihood are used for semiparametric regression modeling and copula estimation

Rank

$$
\begin{aligned}
s_{ij} > s_{ik} \quad &\Rightarrow y_{ij} > y_{ik} \} \; R(S) \\
s_{ij} = 0 \text{ and } d_i < n \quad &\Rightarrow y_{ij} \leq 0 \\
s_{ij} > 0 \quad &\Rightarrow y_{ij} > 0 \\
s_{ij} = 0 \quad &\Rightarrow y_{ij} < 0
\end{aligned}
$$

- Valid but not fully informative: $F(S) \subsetneq R(S)$
- Variants of this likelihood are used for semiparametric regression modeling and copula estimation
- Cannot estimate "sender" specific effects

Binary

$$
\begin{aligned}
s_{ij} > s_{ik} &\quad \Rightarrow y_{ij} > y_{ik} \\
s_{ij} = 0 \text{ and } d_i < n &\quad \Rightarrow y_{ij} \leq 0 \\
\left.\begin{aligned} s_{ij} > 0 &\quad \Rightarrow y_{ij} > 0 \\ s_{ij} = 0 &\quad \Rightarrow y_{ij} < 0 \end{aligned}\right\} & \; B(S)
\end{aligned}
$$

Binary

$$\left.\begin{aligned} s_{ij} > s_{ik} &\Rightarrow y_{ij} > y_{ik} \\ s_{ij} = 0 \text{ and } d_i < n &\Rightarrow y_{ij} \leq 0 \end{aligned}\right.$$

$$\left.\begin{aligned} s_{ij} > 0 &\Rightarrow y_{ij} > 0 \\ s_{ij} = 0 &\Rightarrow y_{ij} < 0 \end{aligned}\right\} B(S)$$

- Neither fully informative nor valid!

Binary

$$\begin{aligned}
s_{ij} > s_{ik} &\Rightarrow y_{ij} > y_{ik} \\
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\end{aligned}$$

- Neither fully informative nor valid!
- Discards information on the ranks

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\begin{aligned}
s_{ij} > s_{ik} \quad & \Rightarrow y_{ij} > y_{ik} \\
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\end{aligned}
$$

- Neither fully informative nor valid!
- Discards information on the ranks
- Ignores the censoring on the outdegrees

Binary

$$
\left.\begin{aligned}
s_{ij} > s_{ik} &\Rightarrow y_{ij} > y_{ik} \\
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s_{ij} > 0 &\Rightarrow y_{ij} > 0 \\
s_{ij} = 0 &\Rightarrow y_{ij} < 0
\end{aligned}\right\} B(S)
$$

- Neither fully informative nor valid!
- Discards information on the ranks
- Ignores the censoring on the outdegrees
- In particular: $F(S) \not\subset B(S)$

Bayesian estimation

- The FRN, rank and binary likelihoods can be expressed as:

$$L_F(\theta : S) = \Pr(Y \in F(S)|\theta) = \int_{F(S)} dP(Y|\theta)$$

$$L_R(\theta : S) = \Pr(Y \in R(S)|\theta) = \int_{R(S)} dP(Y|\theta)$$

$$L_B(\theta : S) = \Pr(Y \in B(S)|\theta) = \int_{B(S)} dP(Y|\theta)$$

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$$L_B(\theta : S) = \Pr(Y \in B(S)|\theta) = \int_{B(S)} dP(Y|\theta)$$

- Generally intractable

Bayesian estimation

- The FRN, rank and binary likelihoods can be expressed as:

$$
\begin{aligned}
L_F(\theta : S) =& \Pr(Y \in F(S)|\theta) = \int_{F(S)} dP(Y|\theta) \\
L_R(\theta : S) =& \Pr(Y \in R(S)|\theta) = \int_{R(S)} dP(Y|\theta) \\
L_B(\theta : S) =& \Pr(Y \in B(S)|\theta) = \int_{B(S)} dP(Y|\theta)
\end{aligned}
$$

- Generally intractable
- Inference for θ can proceed using a MCMC approximation:

Bayesian estimation

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  - Given the observed ranks S and a prior distribution $p(\theta)$

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- Generally intractable
- Inference for θ can proceed using a MCMC approximation:
  - Given the observed ranks S and a prior distribution $p(\theta)$
  - Generate a Markov chain whose stationary distribution is that of (θ, Y) given $Y \in F(S)$, $R(S)$ or $B(S)$

Bayesian estimation

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- Inference for θ can proceed using a MCMC approximation:
  - Given the observed ranks S and a prior distribution $p(\theta)$
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  - Gibbs: simulate $\theta \sim p(\theta|Y)$ and then given $Y \in F(S)$, $R(S)$ or $B(S)$, for $i \neq j$ simulate $y_{ij} \sim p(y_{ij}|\theta, Y_{-ij})$

Bayesian estimation

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$$L_F(\theta : S) = \Pr(Y \in F(S)|\theta) = \int_{F(S)} dP(Y|\theta)$$

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- Generally intractable
- Inference for θ can proceed using a MCMC approximation:
  - Given the observed ranks S and a prior distribution $p(\theta)$
  - Generate a Markov chain whose stationary distribution is that of (θ, Y) given $Y \in F(S)$, $R(S)$ or $B(S)$
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Bayesian estimation

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- Generally intractable
- Inference for θ can proceed using a MCMC approximation:
  - Given the observed ranks S and a prior distribution $p(\theta)$
  - Generate a Markov chain whose stationary distribution is that of (θ, Y) given $Y \in F(S)$, $R(S)$ or $B(S)$
  - Gibbs: simulate $\theta \sim p(\theta|Y)$ and then given $Y \in F(S)$, $R(S)$ or $B(S)$, for $i \neq j$ simulate $y_{ij} \sim p(y_{ij}|\theta, y_{ji})$
- Allows for imputation of missing s_{ij}

Bayesian Estimation for FRN

Model: $Y \sim p(Y|\theta),\ \theta \in \Theta$

Bayesian Estimation for FRN

Model: $Y \sim p(Y|\theta),\ \theta \in \Theta$

Data: $Y \in F(S)$

Bayesian Estimation for FRN

Model: $Y \sim p(Y|\theta), \ \theta \in \Theta$

Data: $Y \in F(S)$

Estimation: Given $p(\theta), p(\theta|Y \in F(S))$ can be approximated by a Gibbs sampler:

Bayesian Estimation for FRN

Model: $Y \sim p(Y|\theta),\ \theta \in \Theta$

Data: $Y \in F(S)$

Estimation: Given $p(\theta), p(\theta|Y \in F(S))$ can be approximated by a Gibbs sampler:

- Simulate $\theta \sim p(\theta|Y)$.

Bayesian Estimation for FRN

Model: $Y \sim p(Y|\theta),\ \theta \in \Theta$

Data: $Y \in F(S)$

Estimation: Given $p(\theta), p(\theta|Y \in F(S))$ can be approximated by a Gibbs sampler:

- Simulate $\theta \sim p(\theta|Y)$.
- Simulate $y_{ij} \sim p(y_{ij}|\theta, Y_{-ij}, Y \in F(S))$:

Bayesian Estimation for FRN

Model: $Y \sim p(Y|\theta),\ \theta \in \Theta$

Data: $Y \in F(S)$

Estimation: Given $p(\theta), p(\theta|Y \in F(S))$ can be approximated by a Gibbs sampler:

- Simulate $\theta \sim p(\theta|Y)$.
- Simulate $y_{ij} \sim p(y_{ij}|\theta, Y_{-ij}, Y \in F(S))$:
    1. $s_{ij} > 0$: $y_{ij} \sim p(y_{ij}|\theta, Y_{-ij})1_{y_{ij}\in(a,b)}$ where $a = \max(y_{ik} : s_{ik} < s_{ij})$ and $b = \min(y_{ik} : s_{ik} > s_{ij})$.

Bayesian Estimation for FRN

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Data: $Y \in F(S)$

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    2. $s_{ij} = 0$ and $d_i < m$: $y_{ij} \sim p(y_{ij}|Y_{-ij}, \theta)1_{y_{ij}\leq 0}$.

Bayesian Estimation for FRN

Model: $Y \sim p(Y|\theta),\ \theta \in \Theta$

Data: $Y \in F(S)$

Estimation: Given $p(\theta), p(\theta|Y \in F(S))$ can be approximated by a Gibbs sampler:

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- Simulate $y_{ij} \sim p(y_{ij}|\theta, Y_{-ij}, Y \in F(S))$:
  1. $s_{ij} > 0$: $y_{ij} \sim p(y_{ij}|\theta, Y_{-ij})1_{y_{ij} \in (a,b)}$ where $a = \max(y_{ik} : s_{ik} < s_{ij})$ and $b = \min(y_{ik} : s_{ik} > s_{ij})$.
  2. $s_{ij} = 0$ and $d_i < m$: $y_{ij} \sim p(y_{ij}|Y_{-ij}, \theta)1_{y_{ij} \leq 0}$.
  3. $s_{ij} = 0$ and $d_i = m$: $y_{ij} \sim p(y_{ij}|Y_{-ij}, \theta)1_{y_{ij} \leq \min(y_{ik} : s_{ik} > 0)}$

Simulations

Letting θ before represent the parameters in a regression model, we generated Y from the following Social Relations Model:

$$y_{ij} = \beta^t x_{ij} + a_i + b_j + \epsilon_{ij}$$

$$\begin{pmatrix} a_i \\ b_i \end{pmatrix} \overset{\text{iid}}{\sim} \text{normal}\left(0, \Sigma_{ab}\right)$$

$$\begin{pmatrix} \epsilon_{ij} \\ \epsilon_{ji} \end{pmatrix} \overset{\text{iid}}{\sim} \text{normal}\left(0, \sigma^2 \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right)$$

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Mean model: $\beta^t x_{ij} = \beta_0 + \beta_r x_{ir} + \beta_c x_{jc} + \beta_{d_1} x_{ij1} + \beta_{d_2} x_{ij2}$

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- x_{ir}, x_{jc}: individual level variables
- x_{ij1}: pair specific variable
- x_{ij2}: co-membership in a group

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- x_{ir}, x_{jc}: individual level variables
- x_{ij1}: pair specific variable
- x_{ij2}: co-membership in a group

$$\beta_r = \beta_c = \beta_{d1} = \beta_{d2} = 1 \text{ and } \beta_0 = -3.26$$
$$x_{ir}, x_{ic}, x_{ij1} \stackrel{\text{iid}}{\sim} \text{N}\left(0, 1\right) \quad x_{ij2} = z_i z_j / .42 \text{ for } z_i \stackrel{\text{iid}}{\sim} \text{binary}\left(1/2\right)$$

Simulations

Letting θ before represent the parameters in a regression model, we generated Y from the following Social Relations Model:

$$y_{ij} = \beta^t x_{ij} + a_i + b_j + \epsilon_{ij}$$
$$\begin{pmatrix} a_i \\ b_i \end{pmatrix} \overset{\text{iid}}{\sim} \text{normal}\left(0, \Sigma_{ab}\right)$$
$$\begin{pmatrix} \epsilon_{ij} \\ \epsilon_{ji} \end{pmatrix} \overset{\text{iid}}{\sim} \text{normal}\left(0, \sigma^2 \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right)$$

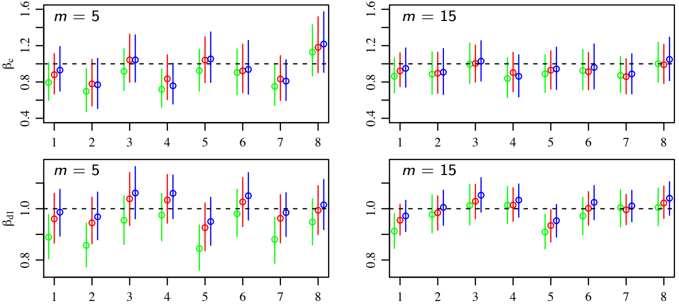
Mean model: $\beta^t x_{ij} = \beta_0 + \beta_r x_{ir} + \beta_c x_{jc} + \beta_{d_1} x_{ij1} + \beta_{d_2} x_{ij2}$

- x_{ir}, x_{jc}: individual level variables
- x_{ij1}: pair specific variable
- x_{ij2}: co-membership in a group

$$\beta_r = \beta_c = \beta_{d1} = \beta_{d2} = 1 \text{ and } \beta_0 = -3.26$$
$$x_{ir}, x_{ic}, x_{ij1} \overset{\text{iid}}{\sim} \mathrm{N}\left(0, 1\right) \quad x_{ij2} = z_i z_j / .42 \text{ for } z_i \overset{\text{iid}}{\sim} \text{binary}\left(1/2\right)$$

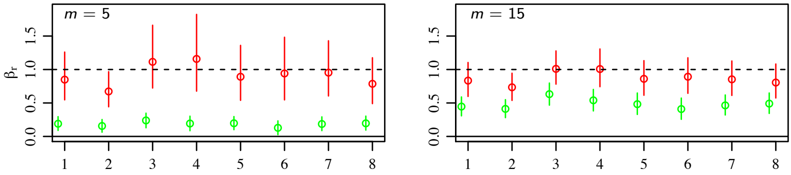
Simulations - Censoring

8 simulations for each $m \in \{5, 15\}$ with 100 nodes each



Confidence intervals under the three different likelihood for column and an iid dyadic variable. The groups of three CIs are based on binary, FRN and rank likelihoods from left to right.

Simulations - Censoring



- Rank likelihood cannot estimate row effects

$$Y \in R(S) \iff Y + c1^t \in R(S)\ \forall c \in \mathbb{R}^m$$

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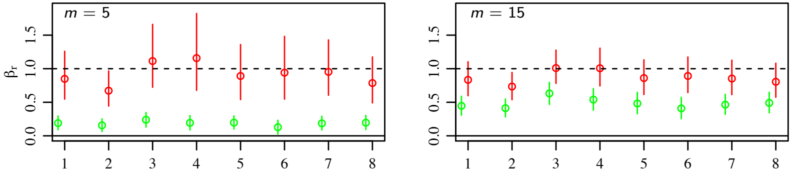

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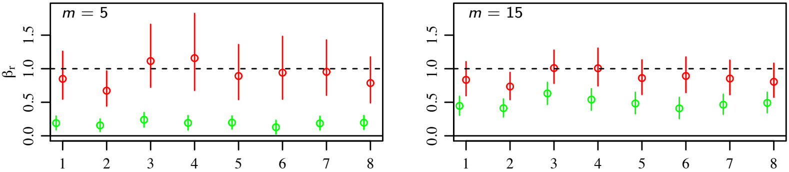

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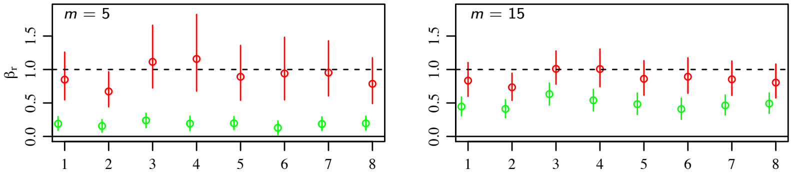

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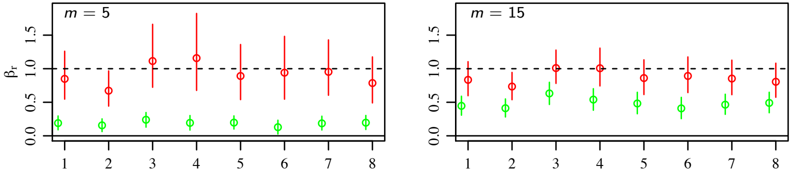

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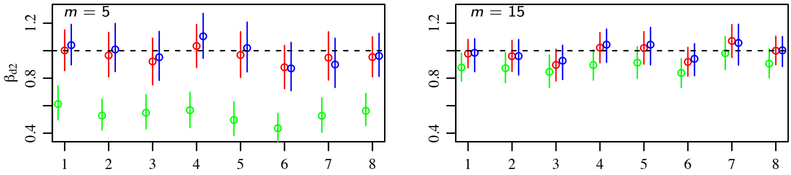
  $\Rightarrow$ Heterogeneity of censored outdegrees is low

  $\Rightarrow$ Regression coefficients estimated too low

Simulations - Censoring



Recall: $x_{ij2} \propto z_i z_j$, an indicator of comembership to a group

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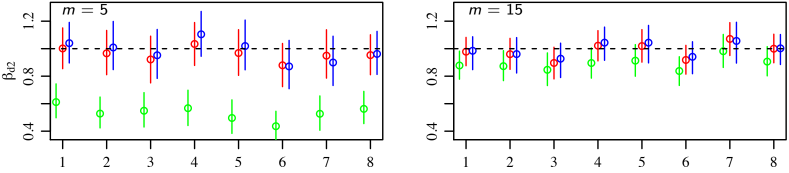

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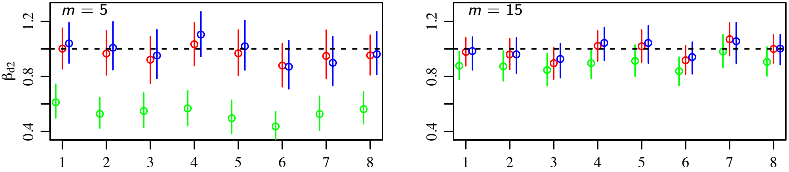

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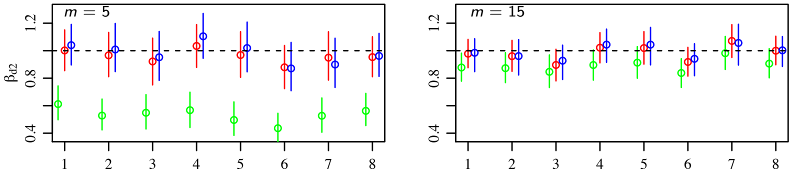

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$\Rightarrow$ Underestimate the variability in x_{ij2}

Simulations - information in the ranks

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Let $C(S)$ be the set of values for which the following is true:

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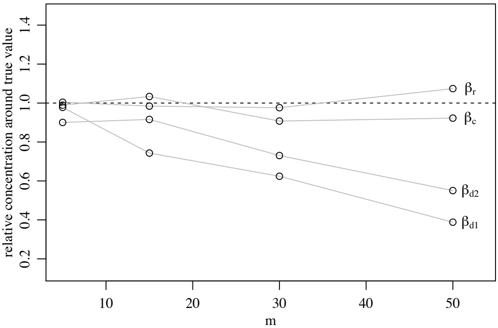
Simulations - information in the ranks

Same setup as before, but average uncensored outdegree is m



Relative concentration around true value of each parameter:
Measured by $\mathrm{E}\left[(\beta-1)^2 \,|F(S)\right] / \mathrm{E}\left[(\beta-1)^2 \,|C(S)\right]$ for each β

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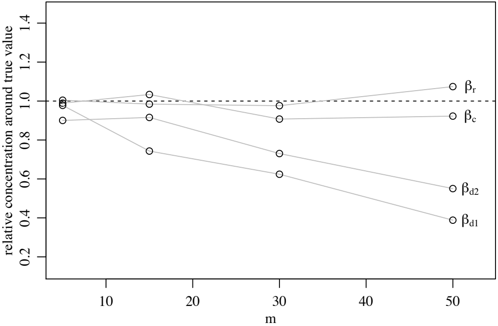

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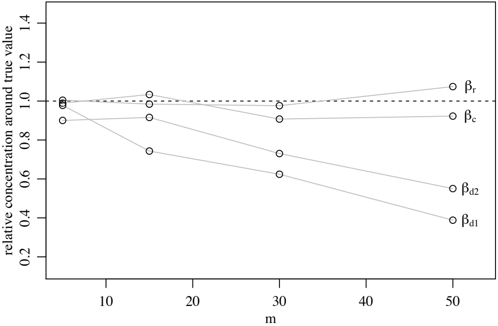

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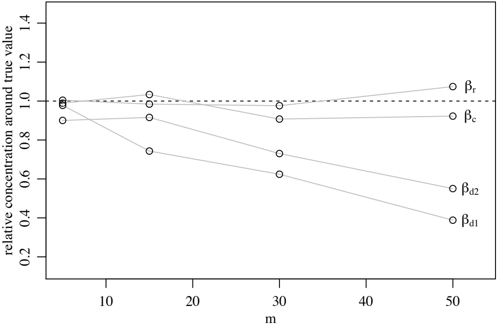

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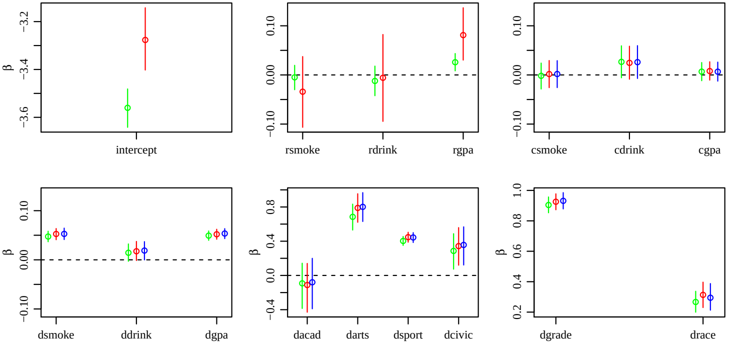
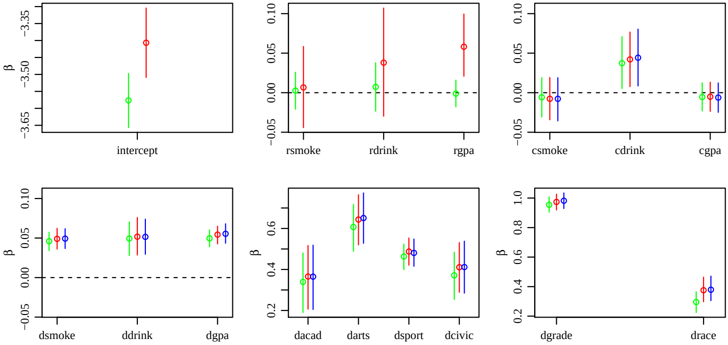
AddHealth Data - Results



- 622 males were asked to rank up to 5 male friends
- Fit a mean model with row, column and dyadic effects for smoking, drinking and gpa as well as dyadic effects for comembership in activities and grade, and a similarity-in-race measure.

AddHealth Data - Results (Females)



Results across schools

Likelihood	intercept	row	column	mean-zero dyadic	other dyadic
binomial	0.89 , 1.68	2.22 , 2.95	1.02 , 1.03	1.06 , 1.06	1.20 , 1.09
rank	NA , NA	NA , NA	1.05 , 0.98	0.99 ,0.99	1.06 , 0.98

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FRN compared to Binary and Rank likelihoods:

- First: Average relative magnitudes of parameter estimates
- Second: Average relative CI widths

  Summary: Under binary:

- $\hat{\beta}_0$ too negative, standard errors too small.
- $|\hat{\beta}_r|$ too small, standard errors too small.

FRN conclusion

- Binary likelihood is likely to underestimate the effects of regressors with variation among the nominators of relations.
- Accounting for censoring is an important first step
- The FRN likelihood can be used in conjunction with latent variable models to capture network features such as transitivity, clustering or stochastic equivalence.