Final Examination

STA 711: Probability & Measure Theory

Saturday, 2015 Dec 12, 2:00 – 5:00 pm

This is a closed-book exam. You may use a sheet of prepared notes, if you wish, but you may not share materials.

If a question seems ambiguous or confusing, *please* ask me to clarify it. Unless a problem states otherwise, you must **show** your **work**. There are blank worksheets at the end of the test if you need more room for this, and also a pdf/pmf sheet.

It is to your advantage to write your solutions as clearly as possible, and to box answers I might not find.

For full credit, answers must be given in **closed form** with no unevaluated sums, integrals, maxima, unreduced fractions. Wherever, possible **simplify**.

Good luck.

1.	/20	5.	/20
2.	/20	6.	/20
3.	/20	7.	/20
4.	/20	8.	/20
	/80		/80
Total:	/160		

Print Name: ______________________________

Problem 1: Let $Z \sim \mathsf{No}(0,1)$ and set $X := Z^2$, $\mathcal{G} := \sigma(Z)$, $\mathcal{H} := \sigma(X)$.

a) (5) Find $a, b \in \mathbb{R}$ such that the random variable $Y := a + bZ$ is the orthogonal projection of X onto the span of Z, *i.e.*, satisfies

$$\mathsf{E}(X-Y)Z = 0 \qquad \text{and} \qquad \mathsf{E}(X-Y)1 = 0$$

$$Y = \underbrace{\_\_\_\_\_\_\_\_}_{a} + \underbrace{\_\_\_\_\_\_\_\_}_{b}\; Z$$

b) (5) Find the conditional expectation of X, given $\mathcal{G} = \sigma(Z)$:

$\mathsf{E}[X \mid \mathcal{G}] =$ ________________

c) (5) Find the conditional expectation of Z, given $\mathcal{H} = \sigma(X)$:

$\mathsf{E}[Z \mid \mathcal{H}] =$ ________________

d) (5) Give an event $A \in \mathcal{G} = \sigma(Z)$ that is not in $\mathcal{H} = \sigma(X)$, if possible, and an event $B \in \mathcal{H}$ that is not in $\mathcal{G}$, if possible. If not possible, explain why.

$\mathcal{G} \backslash \mathcal{H} \ni A =$ ________________ $\qquad$ $\mathcal{H} \backslash \mathcal{G} \ni B =$ ________________

Problem 2: Let $X_n \to X$ *pr.* for some $X \in L_1(\Omega, \mathcal{F}, \mathsf{P})$, and let $Y \in L_2(\Omega, \mathcal{F}, \mathsf{P})$. For each part below answer "Yes" or "No".

If *Yes*, indicate which theorem best justifies your answer by selecting **Fat**ou's Lemma, Lebesgue's **D**ominated or **M**onotone **C**onvergence **T**heorems, the **B**orel/**C**antelli lemma, **Fub**ini's Theorem, or the inequalities of **Jen**sen, **Min**kowski, **Höl**der, or **Mar**kov. No need to show work.

a) If $|X_n|^{1/2} \le Y$ *a.s.*, does $X_n \to X$ in L_1? ◯ No ◯ Yes, by:
◯ Fat ◯ DCT ◯ MCT ◯ B/C ◯ Fub ◯ Jen ◯ Min ◯ Höl ◯ Mar

b) If $X_n \nearrow X \le Y$, does $X_n \to X$ in L_1? ◯ No ◯ Yes, by:
◯ Fat ◯ DCT ◯ MCT ◯ B/C ◯ Fub ◯ Jen ◯ Min ◯ Höl ◯ Mar

c) If $X_n \le Y$, is $\mathsf{E}[X] \ge \limsup \mathsf{E}[X_n]$? ◯ No ◯ Yes, by:
◯ Fat ◯ DCT ◯ MCT ◯ B/C ◯ Fub ◯ Jen ◯ Min ◯ Höl ◯ Mar

d) If $\sum_n \mathbf{1}_{\{X^2 > n\}} < \infty$ *a.s*, is $X \in L_2$? ◯ No ◯ Yes, by:
◯ Fat ◯ DCT ◯ MCT ◯ B/C ◯ Fub ◯ Jen ◯ Min ◯ Höl ◯ Mar

e) If $(\forall \epsilon{>}0) \sum \mathsf{P}[|X_n|{>}\epsilon]{<}\infty$, does $X_n \to 0$ *a.s.*? ◯ No ◯ Yes, by:
◯ Fat ◯ DCT ◯ MCT ◯ B/C ◯ Fub ◯ Jen ◯ Min ◯ Höl ◯ Mar

Problem 3: Let $\Omega := \{0, 1, 2, 3\}$ with probability assignment $\mathsf{P}[E] := \sum_{\omega \in E} 2^\omega/15$ for $E \in \mathcal{F} := 2^\Omega$. Consider events $A := \{0, 1\}$ and $B := \{0, 2\}$, and random variables

$$W(\omega) = \omega \quad X(\omega) = 2^\omega \quad Y(\omega) = \mathbf{1}_A(\omega) \quad Z(\omega) = \mathbf{1}_B(\omega)$$

a) (5) Find the expectation of each RV:

$\mathsf{E}W =$ _________ $\mathsf{E}X =$ _________ $\mathsf{E}Y =$ _________ $\mathsf{E}Z =$ _________

b) (5) Are $\sigma(Y)$ and $\sigma(Z)$ independent? ◯ Yes ◯ No Why?

c) (5) How many events are in the σ-algebra $\sigma(Y, Z)$ generated by Y and Z? You need not enumerate them.

d) (5) Find the conditional expectation $\mathsf{E}[W \mid Y]$.

Problem 4: Let $\{X_n\}$ and Y be real-valued random variables on $(\Omega, \mathcal{F}, \mathsf{P})$ and for $n, k \in \mathbb{N}$ set $A_{n,k} := \big\{\omega : \ |X_n(\omega) - Y(\omega)| > \frac{1}{k}\big\}$.

a) (5) Give the exact conditions on $A_{n,k}$ for $X_n \to Y$ *a.s.*

b) (5) Give the exact conditions on $A_{n,k}$ for $X_n \to Y$ *pr.*

c) (5) Use your expressions above to prove that almost-sure convergence implies convergence in probability.

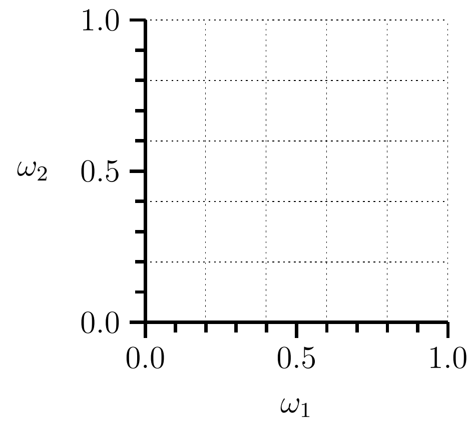
d) (5) Prove that $\sin(X_n) \to \sin(Y)$ in $L_1(\Omega, \mathcal{F}, \mathsf{P})$ if $X_n \to Y$ *a.s.*

Problem 5: Let $\Omega = (0,1]^2 = \{(\omega_1, \omega_2) : \ 0 < \omega_j \le 1\}$ with Lebesgue measure P on the Borel sets $\mathcal{F}$, and consider the random variables

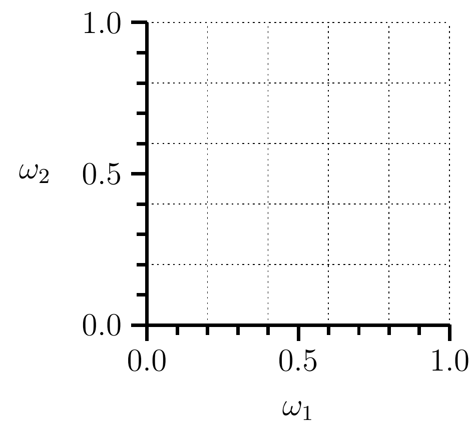
$$X(\omega) := \omega_1 \qquad Y(\omega) := \omega_2 \qquad R(\omega) := \big(\omega_1^2 + \omega_2^2\big)^{1/2} \qquad \Theta(\omega) := \arctan \omega_2/\omega_1$$

(so $\omega_2/\omega_1 = \tan\Theta$, with $0 < \Theta \le \pi/2$)

a) (4) Sketch an event $A \in \sigma(R)$ that is not in $\sigma(X)$, $\sigma(Y)$, or $\sigma(\Theta)$.

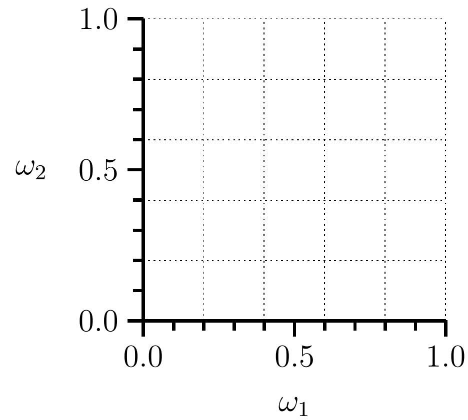


b) (4) Sketch an event $B \in \sigma(\Theta)$ that is not in $\sigma(X)$, $\sigma(Y)$, or $\sigma(R)$.



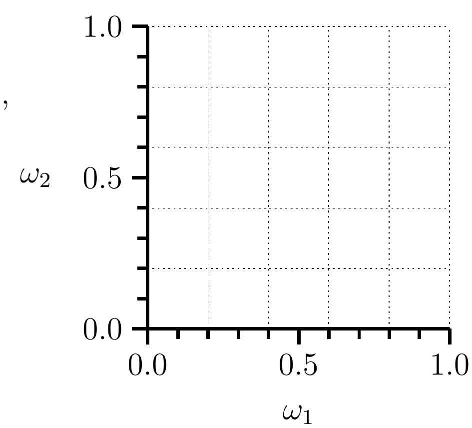
Problem 5 (cont'd): Recall $\Omega = (0,1]^2$ with Lebesgue measure

c) (4) Sketch and label **independent** events $D \in \sigma(R)$ and $E \in \sigma(\Theta)$ that are non-trivial– *i.e.*, have probabilities $0 < \mathsf{P}(D), \mathsf{P}(E) < 1$.



XC) (+2) Are $\sigma(R)$ and $\sigma(\Theta)$ independent? ◯ Yes ◯ No Why?

d) (8) Sketch and label the events $F := \{\omega :\ 0 < \omega_1 \le \omega_2 \le 1\}$ and $G := \{\omega :\ 0 < \omega_1 \le \frac{1}{2},\ 0 < \omega_2 \le 1\}$, and compute $\mathsf{E}[\mathbf{1}_F \mid \mathbf{1}_G]$. No need to prove anything, just compute the conditional expectation.



$\mathsf{E}[\mathbf{1}_F \mid \mathbf{1}_G] =$

Problem 6: Let $\{A_n\}$ be independent events on some probability space $(\Omega, \mathcal{F}, \mathsf{P})$ with $\mathsf{P}(A_n) = 2^{-n}$ for $n \in \mathbb{N}$. Fix $\alpha > 0$ and set

$$X_n := \alpha^n \mathbf{1}_{A_n}.$$

In d) and e), "converge" means "converge to some finite random variable".

a) (4) For which (if any) $\alpha > 0$ is $\{X_n\}$ uniformly bounded in L_1? Why?

b) (4) For which (if any) $\alpha > 0$ is $\{X_n\}$ uniformly bounded in L_4? Why?

c) (4) For which (if any) $\alpha > 0$ is $\{X_n\}$ uniformly bounded in L_∞? Why?

d) (4) For which (if any) $\alpha > 0$ does $\sum_{n\in\mathbb{N}} X_n$ converge *a.s.*? Why?

e) (4) For which (if any) $\alpha > 0$ does $\sum_{n\in\mathbb{N}} X_n$ converge in L_1? Why?

Problem 7: Let $\mathcal{H} \subset \mathcal{G} \subset \mathcal{F}$ be σ-fields on the set $\Omega = (0, 1]$, and let Y be a $\mathcal{G}$-measurable random variable.

a) (10) Does it follow that Y is also $\mathcal{H}$-measurable? ◯ Yes ◯ No
If "Yes", give a proof; if "No", give a counter-example[1].

b) (10) Does it follow that Y is also $\mathcal{F}$-measurable? ◯ Yes ◯ No
If "Yes", give a proof; if "No", give a counter-example[1].

[1]For counter-examples, you might use the σ-algebras $\mathcal{F}_n := \sigma\{(0, i/2^n] : 0 \leq i \leq 2^n\}$

Problem 8: Let $\{X, Y, Z\}$ and $\{X_n\}$ be RVs on the same probability space $(\Omega, \mathcal{F}, \mathsf{P})$. Choose True or False below; no need to explain (unless you can't resist). Each is 2pt.

a) T F If $\mathsf{E}[\exp(X)] < \infty$ then also $\mathsf{E}|X| < \infty$.

b) T F If $X_n \to X$ *pr.* and if Ω is countable then $X_n \to X$ *a.s.*

c) T F If X, Y are independent and both are in L_2 then the product XY is in L_2 too.

d) T F If $Z = g(X, Y)$ for some Borel function $g : \mathbb{R}^2 \to \mathbb{R}$ then $\sigma(Z) \subseteq \sigma(X, Y)$.

e) T F If the events $A := [X < x]$ and $B := [Y > y]$ are independent for each $x, y \in \mathbb{R}$ then X, Y are independent RVs.

f) T F If $\sigma(X) \perp\!\!\!\perp \sigma(Y)$ and $X, Y \in L_1$ then $\mathsf{E}[X \mid Y]$ is the constant random variable with value $\mathsf{E}[X]$.

g) T F If the sum $\sum_n \mathsf{P}(A_n) < \infty$ for the events $A_n := [|X| > n]$, then $X \in L_1(\Omega, \mathcal{F}, \mathsf{P})$.

h) T F If $X, Y \in L_6$ then $XY \in L_3$ with $\|XY\|_3 \le \|X\|_6 \|Y\|_6$.

i) T F Let $\phi(\omega) := \mathsf{E}[e^{i\omega X}]$ and $\psi(\omega) := \mathsf{E}[e^{i\omega Y}]$ be the ch.f.s for X and Y. Then the ch.f. for $Z := X - Y$ is $\phi(\omega)/\psi(\omega)$ if X, Y are independent.

j) T F If $\mathsf{P}[|X| > t] \ge \mathsf{P}[|Y| > t]$ for each $t > 0$ then $\|X\|_2 \ge \|Y\|_2$.

Blank Worksheet

Another Blank Worksheet

Name	Notation	pdf/pmf	Range	Mean μ	Variance σ^2	
Beta	$\mathsf{Be}(\alpha, \beta)$	$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$	$x \in (0,1)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	
Binomial	$\mathsf{Bi}(n, p)$	$f(x) = \binom{n}{x} p^x q^{(n-x)}$	$x \in 0, \cdots, n$	$n\,p$	$n\,p\,q$	$(q = 1-p)$
Exponential	$\mathsf{Ex}(\lambda)$	$f(x) = \lambda\, e^{-\lambda x}$	$x \in \mathbb{R}_+$	$1/\lambda$	$1/\lambda^2$	
Gamma	$\mathsf{Ga}(\alpha, \lambda)$	$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1}\, e^{-\lambda x}$	$x \in \mathbb{R}_+$	α/λ	α/λ^2	
Geometric	$\mathsf{Ge}(p)$	$f(x) = p\, q^x$	$x \in \mathbb{Z}_+$	q/p	q/p^2	$(q = 1-p)$
		$f(y) = p\, q^{y-1}$	$y \in \{1, \ldots\}$	$1/p$	q/p^2	$(y = x+1)$
HyperGeo.	$\mathsf{HG}(n, A, B)$	$f(x) = \frac{\binom{A}{x}\binom{B}{n-x}}{\binom{A+B}{n}}$	$x \in 0, \cdots, n$	$n\,P$	$n\,P\,(1-P)\frac{N-n}{N-1}$	$(P = \frac{A}{A+B})$
Logistic	$\mathsf{Lo}(\mu, \beta)$	$f(x) = \frac{e^{-(x-\mu)/\beta}}{\beta[1+e^{-(x-\mu)/\beta}]^2}$	$x \in \mathbb{R}$	μ	$\pi^2\beta^2/3$	
Log Normal	$\mathsf{LN}(\mu, \sigma^2)$	$f(x) = \frac{1}{x\sqrt{2\pi\sigma^2}} e^{-(\log x-\mu)^2/2\sigma^2}$	$x \in \mathbb{R}_+$	$e^{\mu+\sigma^2/2}$	$e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right)$	
Neg. Binom.	$\mathsf{NB}(\alpha, p)$	$f(x) = \binom{x+\alpha-1}{x} p^\alpha\, q^x$	$x \in \mathbb{Z}_+$	$\alpha q/p$	$\alpha q/p^2$	$(q = 1-p)$
		$f(y) = \binom{y-1}{y-\alpha} p^\alpha\, q^{y-\alpha}$	$y \in \{\alpha, \ldots\}$	α/p	$\alpha q/p^2$	$(y = x+\alpha)$
Normal	$\mathsf{No}(\mu, \sigma^2)$	$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$	$x \in \mathbb{R}$	μ	σ^2	
Pareto	$\mathsf{Pa}(\alpha, \epsilon)$	$f(x) = (\alpha/\epsilon)(1+x/\epsilon)^{-\alpha-1}$	$x \in \mathbb{R}_+$	$\frac{\epsilon}{\alpha-1}{}^*$	$\frac{\epsilon^2\alpha}{(\alpha-1)^2(\alpha-2)}{}^*$	
		$f(y) = \alpha\,\epsilon^\alpha/y^{\alpha+1}$	$y \in (\epsilon, \infty)$	$\frac{\epsilon\,\alpha}{\alpha-1}{}^*$	$\frac{\epsilon^2\alpha}{(\alpha-1)^2(\alpha-2)}{}^*$	$(y = x+\epsilon)$
Poisson	$\mathsf{Po}(\lambda)$	$f(x) = \frac{\lambda^x}{x!} e^{-\lambda}$	$x \in \mathbb{Z}_+$	λ	λ	
Snedecor F	$F(\nu_1, \nu_2)$	$f(x) = \frac{\Gamma(\frac{\nu_1+\nu_2}{2})(\nu_1/\nu_2)^{\nu_1/2}}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} \times x^{\frac{\nu_1-2}{2}} \left[1+\frac{\nu_1}{\nu_2}\,x\right]^{-\frac{\nu_1+\nu_2}{2}}$	$x \in \mathbb{R}_+$	$\frac{\nu_2}{\nu_2-2}{}^*$	$\left(\frac{\nu_2}{\nu_2-2}\right)^2 \frac{2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)}{}^*$	
Student t	$t(\nu)$	$f(x) = \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}}[1+x^2/\nu]^{-(\nu+1)/2}$	$x \in \mathbb{R}$	0^*	$\nu/(\nu-2)^*$	
Uniform	$\mathsf{Un}(a, b)$	$f(x) = \frac{1}{b-a}$	$x \in (a, b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	
Weibull	$\mathsf{We}(\alpha, \beta)$	$f(x) = \alpha\beta\, x^{\alpha-1}\, e^{-\beta\, x^\alpha}$	$x \in \mathbb{R}_+$	$\frac{\Gamma(1+\alpha^{-1})}{\beta^{1/\alpha}}$	$\frac{\Gamma(1+2/\alpha)-\Gamma^2(1+1/\alpha)}{\beta^{2/\alpha}}$	