Midterm Examination II

STA 711: Probability & Measure Theory

Wednesday, 2016 Nov 16, 1:25 – 2:40pm

This is a closed-book exam. You may use a single sheet of prepared notes, if you wish, but you may not share materials.

If a question seems ambiguous or confusing, *please* ask me to clarify it. Unless a problem states otherwise, you must **show** your **work**. There are blank worksheets at the end of the test if you need more room for this, and also a pdf/pmf sheet.

It is to your advantage to write your solutions as clearly as possible, and to box answers I might not find.

For full credit, answers must be given in **closed form** with no unevaluated sums, integrals, maxima, unreduced fractions. Wherever possible, **Simplify**.

Good luck!

Print Name Clearly: ______________________________

1.	/20
2.	/20
3.	/20
4.	/20
5.	/20
Total:	/100

Problem 1: Let $\{X_n\}_{n\in\mathbb{N}} \overset{\text{ind}}{\sim} \mathsf{Ex}(n^2)$ be independent exponentially-distributerd random variables that satisfy $\mathsf{P}[X_n > x] = \exp(-n^2 x)$ for $x > 0$, and set $S_n := \sum_{m=1}^n X_m$.

a) (5) Does S_n converge in L_1? ◯ Yes ◯ No Why?

b) (5) Does S_n converge in L_2? ◯ Yes ◯ No Why?

c) (5) Does X_n converge in L_∞? ◯ Yes ◯ No Why?

d) (5) Does X_n converge almost-surely? ◯ Yes ◯ No Why?

e) (XC) Does S_n converge almost-surely? ◯ Yes ◯ No Why?

Problem 2: Let $(\Omega, \mathcal{F}, \mathsf{P})$ be $\Omega = (0, 1]$ with the Borel sets and Lebesgue measure, and let $\{U_n\} \overset{\text{iid}}{\sim} \mathsf{Un}(0, 1)$ be iid standard uniform random varaiables on $(\Omega, \mathcal{F}, \mathsf{P})$. In each part below, indicate in which (if any) sense(s) the sequence $\{X_n\}$ converges to zero. No explanations are necessary.

a) (5) $X_n := 2^n \mathbf{1}_{\{0<\omega\le 4^{-n}\}}$: ◯ $a.s.$ ◯ $pr.$ ◯ L_1 ◯ L_2 ◯ L_∞

b) (5) $X_n := \sqrt{n}\mathbf{1}_{\{U_n<1/n^2\}}$: ◯ $a.s.$ ◯ $pr.$ ◯ L_1 ◯ L_2 ◯ L_∞

c) (5) $X_n := U_n^{-1}\mathbf{1}_{\{U_n<1/n\}}$: ◯ $a.s.$ ◯ $pr.$ ◯ L_1 ◯ L_2 ◯ L_∞

d) (5) $X_n := 1/(nU_n)$: ◯ $a.s.$ ◯ $pr.$ ◯ L_1 ◯ L_2 ◯ L_∞

Problem 3: Let $\{U_i\} \overset{\text{iid}}{\sim} \mathsf{Un}(0,1)$ be iid standard Uniform random variables, and set $X_i := U_i^{-1/4}$. Show your work in finding:

a) (4) Show that the pdf for X_i is $f(x) = 4x^{-5}\mathbf{1}_{\{x>1\}}$:

b) (4) Find $\|X_i\|_p$ for every $0 < p \le \infty$.

$\|X_i\|_p =$

c) (4) Set $S_n := \sum_{i \le n} X_i$. Find sequences a_n and b_n, if possible, so that

$$\frac{S_n - a_n}{b_n} \Rightarrow \mathsf{No}(0,1)$$

has approximately a standard Normal distribution for large n. Justify.

$a_n =$ $b_n =$

Problem 3 (cont'd): Still $X_i := U_i^{-1/4}$ for $\{U_i\} \overset{\text{iid}}{\sim} \mathsf{Un}(0,1)$.

d) (4) Set $T_n := \sum_{i \le n} X_i^2$. Find sequences c_n and d_n, if possible, so that

$$\frac{T_n - c_n}{d_n} \Rightarrow \mathsf{No}(0,1)$$

has approximately a standard Normal distribution for large n. Justify.

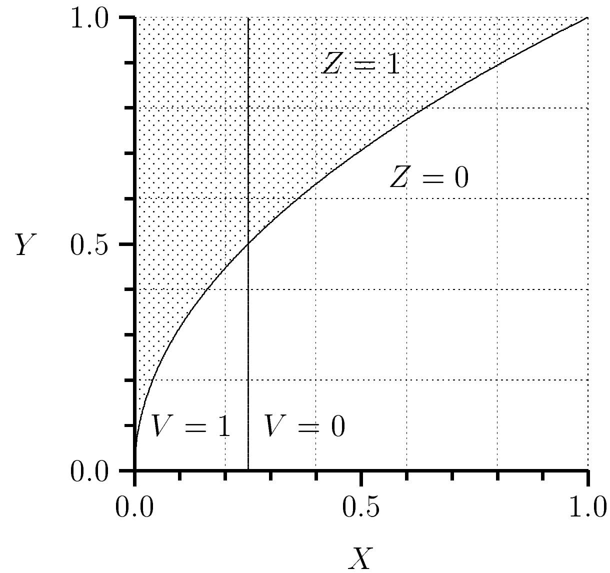
$c_n =$ $\qquad\qquad\qquad$ $d_n =$

e) (4) Find and justify the indicated limits:

$\lim_{n\to\infty} \frac{S_n}{n} =$ $\qquad\qquad\qquad$ $\lim_{n\to\infty} \frac{T_n}{n} =$

Problem 4: The random variables $X, Y \overset{\text{iid}}{\sim} \mathsf{Un}(0,1]$, while

$$V := \mathbf{1}_{\{X \le 0.25\}} \qquad Z := \mathbf{1}_{\{X \le Y^2\}}$$



(and **simplify**!):

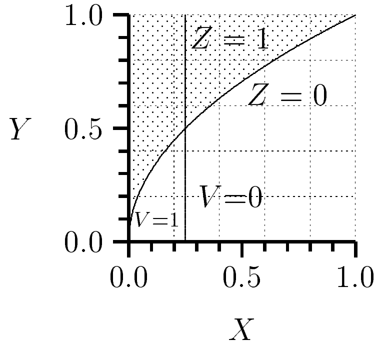
a) (4) Fill in[1] the following table of probabilities:

	$V=0$	$V=1$
$Z=0$		
$Z=1$		

b) (6) $\mathsf{E}[Z \mid V] =$

[1]Remember: Simplify, with no unreduced fractions

Problem 4 (cont'd): Still $X, Y \overset{\text{iid}}{\sim} \mathsf{Un}(0,1]$, $V := \mathbf{1}_{\{X \le 0.25\}}$, $Z := \mathbf{1}_{\{X \le Y^2\}}$.



Find:

c) (5) $\mathsf{E}[X \mid V] =$

d) (5) $\mathsf{E}[Z \mid X] =$

Problem 5: True or false? Circle one; each answer is worth 2 points. No explanations are needed, but you can give one if you think a question seems ambiguous or tricky. All random variables are real on some $(\Omega, \mathcal{F}, \mathsf{P})$.

a) T F If $\{X, Y, Z\}$ are iid & positive then $W := \frac{Z}{X+Y+Z} \in L_1$ and $\mathsf{E}W = 1/3$.

b) T F If X has a continuous pdf with $f(0) > 0$ then $\frac{1}{X} \notin L_1$.

c) T F If $\{X_n\}$ are iid and L_∞ with mean $\mu = \mathsf{E}X_n$ then for some $c > 0$ and all $\epsilon > 0$ and $n \in \mathbb{N}$, $\mathsf{P}[(\bar{X}_n - \mu) > \epsilon] \le \exp(-c\, n\, \epsilon^2)$.

d) T F If $\|X_n\|_2 \to 0$ then also $\mathsf{E}\sqrt{|X_n|} \to 0$.

e) T F For the Cauchy distribution, $\mathsf{E}[\exp(i\omega X)]$ is infinite for all $\omega \in \mathbb{R}$ except for $\omega = 0$ because the Cauchy pdf has heavy tails.

f) T F If $\mathcal{G} \subset \mathcal{F}$ and $Y = \mathsf{E}[X \mid \mathcal{G}]$ with $X \in L_1$, then $\mathsf{E}[X] = \mathsf{E}[Y]$.

g) T F If $\mathcal{G} \subset \mathcal{F}$ and $Y = \mathsf{E}[X \mid \mathcal{G}]$ with $X \in L_2$, then $\mathsf{V}[X] = \mathsf{V}[Y]$.

h) T F Every ch.f. $\phi(\omega) = \mathsf{E}\left[e^{i\omega X}\right]$ is continuously differentiable.

i) T F X and $(-X)$ have the same distribution if and only if $(\forall\omega)\phi_X(\omega) = \phi_X(-\omega)$.

j) T F If $\{X_i\}$ are iid with ch.f. $\phi(\omega)$, then $\prod_{i=1}^n X_i$ has ch.f. $\phi(\omega)^n$.

Blank Worksheet

Another Blank Worksheet

Name	Notation	pdf/pmf	Range	Mean μ	Variance σ^2	
Beta	$\mathsf{Be}(\alpha,\beta)$	$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$x\in(0,1)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	
Binomial	$\mathsf{Bi}(n,p)$	$f(x)=\binom{n}{x}p^x q^{(n-x)}$	$x\in 0,\cdots,n$	$n\,p$	$n\,p\,q$	$(q=1-p)$
Exponential	$\mathsf{Ex}(\lambda)$	$f(x)=\lambda\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	$1/\lambda$	$1/\lambda^2$	
Gamma	$\mathsf{Ga}(\alpha,\lambda)$	$f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	α/λ	α/λ^2	
Geometric	$\mathsf{Ge}(p)$	$f(x)=p\,q^x$	$x\in\mathbb{Z}_+$	q/p	q/p^2	$(q=1-p)$
		$f(y)=p\,q^{y-1}$	$y\in\{1,...\}$	$1/p$	q/p^2	$(y=x+1)$
HyperGeo.	$\mathsf{HG}(n,A,B)$	$f(x)=\frac{\binom{A}{x}\binom{B}{n-x}}{\binom{A+B}{n}}$	$x\in 0,\cdots,n$	$n\,P$	$n\,P\,(1-P)\frac{N-n}{N-1}$	$(P=\frac{A}{A+B})$
Logistic	$\mathsf{Lo}(\mu,\beta)$	$f(x)=\frac{e^{-(x-\mu)/\beta}}{\beta[1+e^{-(x-\mu)/\beta}]^2}$	$x\in\mathbb{R}$	μ	$\pi^2\beta^2/3$	
Log Normal	$\mathsf{LN}(\mu,\sigma^2)$	$f(x)=\frac{1}{x\sqrt{2\pi\sigma^2}}e^{-(\log x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}_+$	$e^{\mu+\sigma^2/2}$	$e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right)$	
Neg. Binom.	$\mathsf{NB}(\alpha,p)$	$f(x)=\binom{x+\alpha-1}{x}p^\alpha\,q^x$	$x\in\mathbb{Z}_+$	$\alpha q/p$	$\alpha q/p^2$	$(q=1-p)$
		$f(y)=\binom{y-1}{y-\alpha}p^\alpha\,q^{y-\alpha}$	$y\in\{\alpha,...\}$	α/p	$\alpha q/p^2$	$(y=x+\alpha)$
Normal	$\mathsf{No}(\mu,\sigma^2)$	$f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}$	μ	σ^2	
Pareto	$\mathsf{Pa}(\alpha,\epsilon)$	$f(x)=(\alpha/\epsilon)(1+x/\epsilon)^{-\alpha-1}$	$x\in\mathbb{R}_+$	$\frac{\epsilon}{\alpha-1}^*$	$\frac{\epsilon^2\alpha}{(\alpha-1)^2(\alpha-2)}^*$	
		$f(y)=\alpha\,\epsilon^\alpha/y^{\alpha+1}$	$y\in(\epsilon,\infty)$	$\frac{\epsilon\,\alpha}{\alpha-1}^*$	$\frac{\epsilon^2\alpha}{(\alpha-1)^2(\alpha-2)}^*$	$(y=x+\epsilon)$
Poisson	$\mathsf{Po}(\lambda)$	$f(x)=\frac{\lambda^x}{x!}e^{-\lambda}$	$x\in\mathbb{Z}_+$	λ	λ	
Snedecor F	$F(\nu_1,\nu_2)$	$f(x)=\frac{\Gamma(\frac{\nu_1+\nu_2}{2})(\nu_1/\nu_2)^{\nu_1/2}}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})}\times$ $x^{\frac{\nu_1-2}{2}}\left[1+\frac{\nu_1}{\nu_2}x\right]^{-\frac{\nu_1+\nu_2}{2}}$	$x\in\mathbb{R}_+$	$\frac{\nu_2}{\nu_2-2}^*$	$\left(\frac{\nu_2}{\nu_2-2}\right)^2\frac{2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)}^*$	
Student t	$t(\nu)$	$f(x)=\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}}[1+x^2/\nu]^{-(\nu+1)/2}$	$x\in\mathbb{R}$	0^*	$\nu/(\nu-2)^*$	
Uniform	$\mathsf{Un}(a,b)$	$f(x)=\frac{1}{b-a}$	$x\in(a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	
Weibull	$\mathsf{We}(\alpha,\beta)$	$f(x)=\alpha\beta\,x^{\alpha-1}\,e^{-\beta\,x^\alpha}$	$x\in\mathbb{R}_+$	$\frac{\Gamma(1+\alpha^{-1})}{\beta^{1/\alpha}}$	$\frac{\Gamma(1+2/\alpha)-\Gamma^2(1+1/\alpha)}{\beta^{2/\alpha}}$	