Midterm Examination I

STA 711: Probability & Measure Theory

Wednesday, 2018 Oct 03, 1:25 – 2:40pm

This is a closed-book exam. You may use a single sheet of prepared notes, if you wish, but you may not share materials.

If a question seems ambiguous or confusing, *please* ask me to clarify it. Unless a problem states otherwise, you must **show** your **work**. There are blank worksheets at the end of the test if you need more room for this, and also a pdf/pmf sheet.

It is to your advantage to write your solutions as clearly as possible, and to box answers I might not find.

For full credit, answers must be given in **closed form** with no unevaluated sums, integrals, maxima, unreduced fractions. Wherever possible, **Simplify**.

Good luck!

Print Name Clearly: ______________________________

1.	/20
2.	/20
3.	/20
4.	/20
5.	/20
Total:	/100

Version a

Problem 1: Let $\Omega = (0, 1]$, $\mathcal{F} = \mathcal{B}(\Omega)$, and $\mathsf{P} = \lambda$ (Lebesgue measure), with random variables

$$X_n(\omega) := \sqrt{n}\mathbf{1}_{\{\omega<1/n\}} \qquad\qquad Y_n(\omega) := \frac{1}{2\sqrt{n\omega}}$$

a) (6) Find the indicated expectations (simplify!):

$\mathsf{E}[X_n] =$ ____________________ $\mathsf{E}[Y_n] =$ ____________________

b) (8) Prove that for each ω, $X_n \to 0$ and $Y_n \to 0$, as follows. For each $0 < \epsilon < 1$, find the smallest $N_\epsilon(\omega)$ such that:

$n \geq N_\epsilon \Rightarrow |X_n(\omega)| \leq \epsilon :$ $N_\epsilon(\omega) =$ ____________________

$n \geq N_\epsilon \Rightarrow |Y_n(\omega)| \leq \epsilon :$ $N_\epsilon(\omega) =$ ____________________

Problem 1 (cont'd): Still $\Omega = (0, 1]$, $\mathcal{F} = \mathcal{B}(\Omega)$, $\mathsf{P} = \lambda$, and

$$X_n(\omega) := \sqrt{n}\mathbf{1}_{\{\omega<1/n\}}, \qquad Y_n(\omega) := \frac{1}{2\sqrt{n\omega}}$$

c) (6) For each $n \in \mathbb{N}$, find the indicated probabilities:

$\mathsf{P}[X_n \geq 10] =$ ____________________

$\mathsf{P}[Y_n \geq 10] =$ ____________________

$\mathsf{P}[Y_n \geq X_n] =$ ____________________

Problem 2: Let $\Omega = (0,1]$, $\mathcal{F} = \mathcal{B}(\Omega)$, and $\mathsf{P} = \lambda$ (Lebesgue measure), with random variables

$$X_n(\omega) := \sqrt{n}\mathbf{1}_{\{\omega<1/n\}} \qquad\qquad Y_n(\omega) := \frac{1}{2\sqrt{n\omega}}$$

a) (6) For which $0 < p < \infty$ is X_n in L_p? How about Y_n?
X_n :

Y_n :

b) (8) Does the Dominated Convergence Theorem apply to X_n and Y_n? If so, find a dominating RV $Z \in L_1$; if not, explain why.
X_n : ◯ Yes ◯ No $Z =$

Y_n : ◯ Yes ◯ No $Z =$

c) (6) Does the Monotone Convergence Theorem apply to X_n and Y_n?
X_n : ◯ Yes ◯ No Why?

Y_n : ◯ Yes ◯ No Why?

Problem 3: Let $\Omega := \{a, b, c, d\}$ with $\mathcal{F} := 2^\Omega$ and P that assigns probabilities 0.20, 0.40, and 0.10 respectively to the singleton sets $\{a\}$, $\{b\}$, $\{c\}$. Let X and Y be RVs given by the following table

	a	b	c	d
$X:$	5	0	1	1
$Y:$	7	2	7	2

a) (8) Are X and Y independent?
Y N Why?

b) (6) Give the σ-algebras $\sigma(X)$ and $\sigma(Y)$ explicitly, by listing their members (no explanations needed):
$\sigma(X) =$

$\sigma(Y) =$

c) (6) Describe the σ-algebra $\sigma(Z)$ for the RV $Z := X + Y$.
Justify your answer.

Problem 4: Let $(\Omega, \mathcal{F}, \mathsf{P})$ be the nonnegative integers $\Omega = \mathbb{N} := \{1, 2, 3, \dots\}$ with $\mathcal{F} = 2^\Omega$ and $\mathsf{P}[A] := \frac{90}{\pi^4} \sum_{\omega \in A} \omega^{-4}$ for $A \in \mathcal{F}$ (see footnote[1]).

a) (2) Show that for any positive decreasing function $\phi : \mathbb{R} \to \mathbb{R}_+$,

$$\sum_{n=2}^{\infty} \phi(n) \le \int_1^\infty \phi(x)\,dx \le \sum_{n=1}^{\infty} \phi(n).$$

b) (6) For $p > 0$, is the random variable $X(\omega) := \omega$ in $L_p(\Omega, \mathcal{F}, \mathsf{P})$? If this depends on p, explain.

◯ Yes ◯ No ◯ It Depends Reasoning?

$p \in$ ________________

c) (XC) If so, give an explicit upper bound for $\|X\|_p$.

$\|X\|_p \le$ ________________

[1]Recall that $\zeta(2) = \sum_{n=1}^\infty n^{-2} = \pi^2/6$ and $\zeta(4) = \sum_{n=1}^\infty n^{-4} = \pi^4/90$

Problem 4 (cont'd): Still $\Omega = \mathbb{N}$, $\mathcal{F} = 2^\Omega$, and $\mathsf{P}[A] := \frac{90}{\pi^4} \sum_{\omega \in A} \omega^{-4}$.

d) (6) For $n \in \mathbb{N}$ set $Y_n(\omega) := \omega^3 \mathbf{1}_{\{\omega \le n\}}$. Does the Dominated Convergence Theorem apply to $\{Y_n\}$? If so, tell what L_1 function Y dominates $\{Y_n\}$; if not, explain why. ◯ Yes ◯ No $Y(\omega) =$
Reasoning?

e) (6) For $n \in \mathbb{N}$ set $Z_n(\omega) := \omega^2 \mathbf{1}_{\{\omega \le n\}}$. Does the Monotone Convergence Theorem apply to $\{Z_n\}$? If so, tell what MCT says and show why it applies; if not, explain why. ◯ Yes ◯ No Reasoning:

Problem 5: True or false? Circle one; each answer is worth 2 points. No explanations are needed, but you can give one if you think the question is ambiguous. All random variables are real on some $(\Omega, \mathcal{F}, \mathsf{P})$.

a) T F If $A \subsetneq B$ (so $B \setminus A \neq \emptyset$) then $\mathsf{P}[A] < \mathsf{P}[B]$.

b) T F If $\mathsf{P}[A] > \mathsf{P}[B] > \mathsf{P}[A \cap B] > 0$ then $\mathsf{P}[A \mid B] > \mathsf{P}[B \mid A]$.

c) T F If $\{X, Y, Z\}$ are iid and $\mathsf{P}[X < Y < Z] = 1/6$ then X has a continuous distribution.

d) T F If $\mathsf{P}[Z > 0] = 1$ then $\mathsf{E}[Z] \cdot \mathsf{E}[1/Z] \geq 1$.

e) T F Every σ-algebra is a π-system.

f) T F If $\mathsf{P}[X_n \to X] = 1$ then $\cos(X_n) \to \cos(X)$ in L_2.

g) T F If $\mathsf{P}[X_n \to X] = 1$ and if $g : \mathbb{R} \to \mathbb{R}$ is Borel, then $\mathsf{P}[g(X_n) \to g(X)] = 1$.

h) T F If $\sum \mathsf{P}[A_n] < \infty$ then $\mathsf{P}[\limsup A_n] = 0$, whether or not $\{A_n\}$ are independent.

i) T F If $\{X_\alpha\}$ are independent and $\{g_\alpha\}$ are Borel then $\{g_\alpha(X_\alpha)\}$ are independent too.

j) T F If $\mathsf{E}|X|^p < \infty$ for all $p > 0$ then $X \in L_\infty$.

Blank Worksheet

Another Blank Worksheet

Name	Notation	pdf/pmf	Range	Mean μ	Variance σ^2	
Beta	$\mathsf{Be}(\alpha,\beta)$	$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$x\in(0,1)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	
Binomial	$\mathsf{Bi}(n,p)$	$f(x)=\binom{n}{x}p^x q^{(n-x)}$	$x\in 0,\cdots,n$	$n\,p$	$n\,p\,q$	$(q=1-p)$
Exponential	$\mathsf{Ex}(\lambda)$	$f(x)=\lambda\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	$1/\lambda$	$1/\lambda^2$	
Gamma	$\mathsf{Ga}(\alpha,\lambda)$	$f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	α/λ	α/λ^2	
Geometric	$\mathsf{Ge}(p)$	$f(x)=p\,q^x$	$x\in\mathbb{Z}_+$	q/p	q/p^2	$(q=1-p)$
		$f(y)=p\,q^{y-1}$	$y\in\{1,...\}$	$1/p$	q/p^2	$(y=x+1)$
HyperGeo.	$\mathsf{HG}(n,A,B)$	$f(x)=\frac{\binom{A}{x}\binom{B}{n-x}}{\binom{A+B}{n}}$	$x\in 0,\cdots,n$	$n\,P$	$n\,P\,(1-P)\frac{N-n}{N-1}$	$(P=\frac{A}{A+B})$
Logistic	$\mathsf{Lo}(\mu,\beta)$	$f(x)=\frac{e^{-(x-\mu)/\beta}}{\beta[1+e^{-(x-\mu)/\beta}]^2}$	$x\in\mathbb{R}$	μ	$\pi^2\beta^2/3$	
Log Normal	$\mathsf{LN}(\mu,\sigma^2)$	$f(x)=\frac{1}{x\sqrt{2\pi\sigma^2}}e^{-(\log x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}_+$	$e^{\mu+\sigma^2/2}$	$e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right)$	
Neg. Binom.	$\mathsf{NB}(\alpha,p)$	$f(x)=\binom{x+\alpha-1}{x}p^\alpha\,q^x$	$x\in\mathbb{Z}_+$	$\alpha q/p$	$\alpha q/p^2$	$(q=1-p)$
		$f(y)=\binom{y-1}{y-\alpha}p^\alpha\,q^{y-\alpha}$	$y\in\{\alpha,...\}$	α/p	$\alpha q/p^2$	$(y=x+\alpha)$
Normal	$\mathsf{No}(\mu,\sigma^2)$	$f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}$	μ	σ^2	
Pareto	$\mathsf{Pa}(\alpha,\epsilon)$	$f(x)=(\alpha/\epsilon)(1+x/\epsilon)^{-\alpha-1}$	$x\in\mathbb{R}_+$	$\frac{\epsilon}{\alpha-1}$ if $\alpha>1$	$\frac{\epsilon^2\alpha}{(\alpha-1)^2(\alpha-2)}$ if $\alpha>2$	
		$f(y)=\alpha\,\epsilon^\alpha/y^{\alpha+1}$	$y\in(\epsilon,\infty)$	$\frac{\epsilon\,\alpha}{\alpha-1}$ if $\alpha>1$	$\frac{\epsilon^2\alpha}{(\alpha-1)^2(\alpha-2)}$ if $\alpha>2$	$(y=x+\epsilon)$
Poisson	$\mathsf{Po}(\lambda)$	$f(x)=\frac{\lambda^x}{x!}e^{-\lambda}$	$x\in\mathbb{Z}_+$	λ	λ	
Snedecor F	$F(\nu_1,\nu_2)$	$f(x)=\frac{\Gamma(\frac{\nu_1+\nu_2}{2})(\nu_1/\nu_2)^{\nu_1/2}}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})}\times$ $x^{\frac{\nu_1-2}{2}}\left[1+\frac{\nu_1}{\nu_2}x\right]^{-\frac{\nu_1+\nu_2}{2}}$	$x\in\mathbb{R}_+$	$\frac{\nu_2}{\nu_2-2}$ if $\nu_2>2$	$\left(\frac{\nu_2}{\nu_2-2}\right)^2\frac{2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)}$ if $\nu_2>4$	
Student t	$t(\nu)$	$f(x)=\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}}[1+x^2/\nu]^{-(\nu+1)/2}$	$x\in\mathbb{R}$	0 if $\nu>1$	$\frac{\nu}{\nu-2}$ if $\nu>2$	
Uniform	$\mathsf{Un}(a,b)$	$f(x)=\frac{1}{b-a}$	$x\in(a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	
Weibull	$\mathsf{We}(\alpha,\beta)$	$f(x)=\alpha\beta\,x^{\alpha-1}\,e^{-\beta\,x^\alpha}$	$x\in\mathbb{R}_+$	$\frac{\Gamma(1+\alpha^{-1})}{\beta^{1/\alpha}}$	$\frac{\Gamma(1+2/\alpha)-\Gamma^2(1+1/\alpha)}{\beta^{2/\alpha}}$	