

# Sta 711: Homework 11

## Martingales

A sequence  $\{(X_n, \mathcal{F}_n), n \geq 0\}$  of random variables  $X_n \in L_1(\Omega, \mathcal{F}, \mathbb{P})$  and a nested sequence of  $\sigma$ -algebras  $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots \subset \mathcal{F}$  is a *martingale* if for each  $0 \leq n \leq m < \infty$

$$X_n = \mathbb{E}[X_m \mid \mathcal{F}_n],$$

so on average the sequence neither increases nor decreases. Note this implies that  $X_n$  is  $\mathcal{F}_n$ -measurable. By the tower property of conditional expectations, it is enough to check this condition for  $m = n + 1$ .

1. A sequence  $\{(X_n, \mathcal{F}_n), n \geq 0\}$  is *predictable* if  $X_{n+1}$  is  $\mathcal{F}_n$ -measurable for each  $n$ . Show every predictable martingale is constant (*i.e.*,  $X_n = X_0$  a.s.).
2. A sequence  $\{(X_n, \mathcal{F}_n), n \geq 0\} \subset L_1(\Omega, \mathcal{F}, \mathbb{P})$  is a *submartingale* if, for each  $0 \leq n \leq m < \infty$ ,  $\mathcal{F}_n \subseteq \mathcal{F}_m \subseteq \mathcal{F}$  and  $X_n \leq \mathbb{E}[X_m \mid \mathcal{F}_n]$ — so, on average,  $X_n$  is increasing.  
Let  $\{(X_n, \mathcal{F}_n), n \geq 0\}$  and  $\{(Y_n, \mathcal{F}_n), n \geq 0\}$  be submartingales on  $(\Omega, \mathcal{F}, \mathbb{P})$ . Show that their max  $(X_n \vee Y_n)$  and sum  $(X_n + Y_n)$  are submartingales too.
3. Show that every submartingale  $\{(X_n, \mathcal{F}_n)\}$  can be written as the sum  $X_n = (M_n + A_n)$  of a martingale  $\{(M_n, \mathcal{F}_n)\}$  and a predictable non-decreasing process  $A_n$ , and that the decomposition is unique if we set  $A_0 = 0$ . Suggestion: How must  $A_n$  be defined to make  $A_0 = 0$  and  $\mathbb{E}[(X_{n+1} - A_{n+1}) \mid \mathcal{F}_n] = (X_n - A_n)$ ?
4. Let  $\{(M_n, \mathcal{F}_n)\}$  be a martingale and  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  a convex function for which  $X_n := \phi(M_n)$  is in  $L_1(\Omega, \mathcal{F}, \mathbb{P})$ . Show that  $\{(X_n, \mathcal{F}_n)\}$  is a submartingale.
5. Fix  $0 < p < 1$ , set  $q := 1 - p$ , and let  $\{\xi_j\}$  be iid random variables with  $\mathbb{P}[\xi_j = 1] = p$  and  $\mathbb{P}[\xi_j = -1] = q$ . Set

$$S_n := \sum_{j \leq n} \xi_j,$$

a random walk (possibly an asymmetric one) on the integers starting at  $S_0 = 0$ .

- (a) For which  $\alpha \in \mathbb{R}$  is  $X_n := [S_n - \alpha n]$  a martingale?
- (b) For which  $\alpha, \beta \in \mathbb{R}$  is  $Y_n := [(S_n - \alpha n)^2 - \beta n]$  a martingale?
- (c) For which  $r > 0$  is  $Z_n := [r^{S_n}]$  a martingale?
- (d) Is  $S_n$  a submartingale?

Of course the answers will depend on  $p$ .

## Extremes

Most computations about extremes depend in one way or another on the elementary limit “ $(1 + z/n)^n \rightarrow e^z$ ”—the trick is arranging for  $\mathbb{P}[X_n^* \leq x]$  to look like  $(1 + z/n)^n$ .

6. Let  $\{X_j\}_{j \in \mathbb{N}} \stackrel{\text{iid}}{\sim} \text{Ex}(\lambda)$  be independent exponentially-distributed random variables, and let

$$X_n^* := \max\{X_j : 1 \leq j \leq n\}$$

be the maximum of the first  $n$ . Find sequences  $\{a_n, b_n\}$  of real numbers such that

$$Y_n := \frac{X_n^* - b_n}{a_n} \Rightarrow \text{Gu}(0, 1),$$

the standard Gumbel distribution. The  $\text{Gu}(m, s)$  distribution has CDF

$$F(x) = \exp\left(-e^{-(x-m)/s}\right)$$

for  $x \in \mathbb{R}$ . The maxima from many other distributions (normal, gamma, etc.) are also approximately Gumbel (you don't have to prove that!).

7. The standard Student's  $t$  distribution with  $\nu$  degrees of freedom,  $t_\nu$ , has pdf

$$f_\nu(t) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi} \Gamma\left(\frac{\nu}{2}\right)} (1 + t^2/\nu)^{-(\nu+1)/2}$$

for  $t \in \mathbb{R}$ . Let  $\{T_j\} \stackrel{\text{iid}}{\sim} t_\nu$  and set  $T_n^* := \max\{T_j : 1 \leq j \leq n\}$ , and find sequences  $\{a_n, b_n\}$  of real numbers such that  $(T_n^* - b_n)/a_n$  converges in distribution to some limit. What's the limiting distribution? You don't need to prove it, but this is also the limiting distribution for the maxima of many other heavy-tailed distributions (Pareto,  $\alpha$ -Stable, log Normal, etc.).

Hints: (1) Don't get mesmerized by the normalizing constant, and (2) if  $x$  is huge and  $p$  is any fixed power then  $(x+1)^{-p}$  and  $(x)^{-p}$  don't differ by much