STA 711: Probability and Measure Theory

Analysis & Calculus Quiz

Students in STA 711: Probability & Measure Theory are expected to be familiar with real analysis at an advanced undergraduate level— the level of W. Rudin's *Principles of Mathematical Analysis* or M. Reed's *Fundamental Ideas of Analysis*. They should be able to answer the questions in this quiz without consulting reference materials.

Problem 1: Recall that a sequence $\{x_n\}$ in a metric space $(\mathcal{X}, \mathrm{d})$ *converges* to a limit $x^* \in \mathcal{X}$ if for each $\epsilon > 0$ there exists a number $N_\epsilon < \infty$ such that

$$(\forall n \geq N_\epsilon) \quad \mathrm{d}(x_n, x^*) < \epsilon.$$

a. Prove[1] that $x_n := 1/\sqrt{n}$ converges to $x^* = 0$ in the metric space $\mathcal{X} = \mathbb{R}$ with the usual (Euclidean) distance metric $\mathrm{d}(x, y) := |x - y| = \sqrt{(x-y)^2}$.

b. Find an explicit sequence x_n of rational numbers that converges to $x^* = \pi$ in the metric space $\mathcal{X} = \mathbb{R}$. Prove that it converges, by finding N_ϵ (Hint: you might want to *start* by choosing N_ϵ— say, $\lceil 1/\epsilon \rceil$ or $\lceil -\log_2 \epsilon \rceil$ or $\lceil -\log_{10} \epsilon \rceil$— and then find x_n).

[1]Find N_ϵ explicitly. You may find the function $\lfloor x \rfloor := \max\{k \in \mathbb{Z} : \ k \leq x\}$ (the greatest integer less than or equal to x) to be useful, or perhaps $\lceil x \rceil := \min\{k \in \mathbb{Z} : \ k \geq x\}$.

Problem 2: Recall that a subset E of a metric space $(\mathcal{X}, \mathrm{d})$ is *open* if for each $x \in E$ there exists some $\epsilon_x > 0$ such that the entire ball

$$B_\epsilon(x) = \{\xi \in \mathcal{X} :\ \mathrm{d}(x, \xi) < \epsilon_x\} \subset E$$

lies within E. A set $F \subset \mathcal{X}$ is *closed* if its complement $F^c = \{x \in \mathcal{X} :\ x \notin F\}$ is open.

a. Prove that $(0, 1)$ is open in $\mathcal{X} = \mathbb{R}$.

b. Prove that any union $U = \cup E_\alpha$ of open sets is also open.

c. Show by example that the union $U = \cup F_\alpha$ of closed sets may not be closed.

Problem 3: Recall that a set K in a metric space $(\mathcal{X}, \mathrm{d})$ is *compact*[2] if every open cover $K \subset \cup_\alpha U_\alpha$ admits a finite sub-cover $K \subset \cup_{i=1}^n U_{\alpha_i}$, and that a function $f(\cdot) : \mathcal{X} \to \mathcal{Y}$ from one metric space to another is *continuous* if for every open set $U \subset \mathcal{Y}$, $f^{-1}(U) := \{x : \; f(x) \in U\}$ is an open set in $\mathcal{X}$.

a. Prove that every *compact* set K is also *closed*.

b. If K is a *compact* set and $F \subset K$ is a *closed* subset, prove that F is also compact.

c. If $f : \mathcal{X} \to \mathbb{R}$ is a *continuous* real-valued function and $K \subset \mathcal{X}$ is compact, prove that the supremum

$$M := \sup_{x \in K} f(x)$$

is finite.

d. Show[3] this can fail if f is not continuous— *i.e.*, give an example of an unbounded (but finite) function f on a compact set K.

[2]The Heine-Borel theorem says *in Euclidean space* any closed & bounded set is compact, but that doesn't hold in general. For example, the unit ball $B := \{f : \int_0^1 |f(x)|^2 \, dx \le 1\}$ is closed and bounded in $L_2\big((0,1]\big)$ but is not compact, since the sequence of functions $\{f_n(x) := \sqrt{2}\sin(n\pi x)\} \subset B$ has no limit point in B.

[3]Suggestion: take $K = [0,1]$ on $\mathcal{X} = \mathbb{R}$, and define $f(x)$ by cases. What cases?

Problem 4:

a. Let K_α be compact for each index α and suppose that each *finite* intersection $\cap_{j=1}^n K_{\alpha_j} \neq \emptyset$ is non-empty. Prove that $\cap_\alpha K_\alpha \neq \emptyset$.

b. If $f : \mathcal{X} \to \mathbb{R}$ is real-valued and continuous with supremum $M := \sup_{x \in K} f(x)$ on a compact set $K \subset \mathcal{X}$, prove that there exists some $x^* \in K$ for which $f(x^*) = M$— *i.e.*, that the supremum is attained.

Problem 5:

a. Give an example of a closed set $C \subset \mathbb{R}$ that is *not* compact.

b. Give an example of a set $A \subset \mathbb{R}$ that is neither closed nor open.

c. Give an example of a set $B \subset \mathbb{R}$ that is both closed and open.

Problem 6: Evaluate the sums and integrals below for *every* value of $p \in \mathbb{R}$ (some expressions might be infinite or undefined for some values of p):

a. $\displaystyle\int_0^1 x^p \, dx =$

b. $\displaystyle\int_1^\infty x^p \, dx =$

c. $\displaystyle\int_0^\infty e^{-px} \, dx =$

d. $\displaystyle\sum_{n=2}^{9} p^n =$

e. $\displaystyle\sum_{n=1}^{\infty} p^n =$

f. $\displaystyle\sum_{n=7}^{\infty} n\, p^n =$

g. $\displaystyle\int_0^\infty x\, e^{-px^2} \, dx =$

h. $\displaystyle\int_0^x \sin(\ln u) \, du =$

i. $\displaystyle\int_0^\pi e^{-p\cos(x)} \sin(x) \, dx =$

Problem 7: Which of the following sums and integrals converges (to a finite limit)? Why? You need not evaluate the limit.

a. T F $\displaystyle\int_2^\infty \frac{\ln(e^x - 2)}{x^3 + 1}\, dx$ converges:

b. T F $\displaystyle\sum_{n=0}^\infty \frac{3^n (n!)^2}{(2n)!}$ converges:

c. T F $\displaystyle\sum_{n=1}^\infty \frac{\ln n + \sin n}{n^{3/2}}$ converges:

d. T F $\displaystyle\int_0^\infty \frac{\sin x}{x^{3/2}}\, dx$ converges:

e. T F $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x} + x^2}$ converges:

f. T F $\displaystyle\int_0^1 \frac{\tan x}{x^2}\, dx$ converges: