Final Examination

STA 711: Probability & Measure Theory

Saturday, 2019 Dec 14, 2:00 – 5:00 pm

This is a closed-book exam. You may use a sheet of prepared notes, if you wish, but you may not share materials.

If a question seems ambiguous or confusing, *please* ask me to clarify it. Unless a problem states otherwise, you must **show** your **work**. There are blank worksheets at the end of the test if you need more room for this, and also a pdf/pmf sheet.

It is to your advantage to write your solutions as clearly as possible, and to box answers I might not find.

For full credit, answers must be given in **closed form** with no unevaluated sums, integrals, maxima, unreduced fractions. Wherever, possible **simplify**.

Good luck.

1.	/20	5.	/20
2.	/20	6.	/20
3.	/20	7.	/20
4.	/20	8.	/20
	/80		/80
Total:	/160		

Print Name: ______________________

Problem 1: A few questions about $L_p(\Omega, \mathcal{F}, \mathsf{P})$ and convergence:

a) (5) Suppose that the sequence $\{X_n\}$ converges both *a.s.* and in L_2—say, $X_n \to Y_1$ (*a.s.*), and $X_n \to Y_2$ in L_2. Prove $\mathsf{P}[Y_1 = Y_2] = 1$.

b) (5) Suppose X_1, X_2, X_3 are all RVs on $(\Omega, \mathcal{F}, \mathsf{P})$ with

$$\|X_1\|_1 = 1 \qquad \|X_2\|_2 = 2 \qquad \|X_3\|_3 = 3$$

Set $S := X_1 + X_2 + X_3$. For what $p > 0$ is $S \in L_p$?
Give and justify a bound for $\|S\|_p$: $\quad \|S\|_p \leq$

Problem 1 (cont'd): More about $L_p(\Omega, \mathcal{F}, \mathsf{P})$

c) (5) Fix $p > 1$. Let $A \in \mathcal{F}$ be an event with probability $a := \mathsf{P}[A]$ and let $X \in L_p(\Omega, \mathcal{F}, \mathsf{P})$ be a positive RV with norm $x := \|X\|_p$. Find and justify a non-trivial upper bound (one that tends to zero as $a \to 0$ or $x \to 0$):
$\mathsf{E}\big[\,|X\mathbf{1}_A|\,\big] \le$

d) (5) Let $\{X, Y\} \subset L_1$ be independent and non-negative, but *not* in L_2. Prove that $XY \in L_1$. Suggestion: For $t > 0$, consider $(X \wedge t)(Y \wedge t)$, where $x \wedge y := \min(x, y)$ for $x, y \in \mathbb{R}$. Then what?

Problem 2: Let $\Omega = \mathbb{R}$ and $\mathcal{F} = \mathcal{B}(\Omega)$, the Borel sets on $\mathbb{R}$. Define two probability meaasures on $(\Omega, \mathcal{F})$ by

$$\mathsf{P}(d\omega) := \tfrac{1}{2} e^{-|\omega|}\, d\omega \qquad\qquad \mathsf{Q}(d\omega) := \mathbf{1}_{\{\omega>0\}} e^{-\omega}\, d\omega.$$

Set $\mathcal{B} := \{(-a, a) : \ a \in \mathbb{R}_+\}$ and $\mathcal{G} := \sigma(\mathcal{B})$.

a) (5) Prove that P and Q agree on all of $\mathcal{G}$.

b) (5) Prove that P and Q do *not* agree on all of $\mathcal{F}$.

Problem 2 (cont'd): Still $\mathsf{P}(d\omega) := \frac{1}{2}e^{-|\omega|}\,d\omega$ and $\mathsf{Q}(d\omega) := \mathbf{1}_{\{\omega>0\}}e^{-\omega}\,d\omega$, with $\mathcal{B} := \{(-a,a):\ a \in \mathbb{R}_+\}$ and $\mathcal{G} := \sigma(\mathcal{B})$.

c) (5) Give an example of a non-constant $\mathcal{G}$-measurable RV X.
$X(\omega) =$

d) (5) Let $\{X_n\}$ be iid RVs with distribution P. Find a non-trivial DF G such that the maximum $X_n^* := \max\{X_j:\ 1 \le j \le n\}$ satisfies

$$(\forall x \in \mathbb{R}) \qquad \mathsf{P}\Big[(X_n^* - \log n) \le x\Big] \to G(x)$$

Problem 3: Let $\{A_n\} \subset \mathcal{F}$ and $\{X_n\} \subset L_1(\Omega, \mathcal{F}, \mathsf{P})$ with $\|X_n\|_1 \le 1$ and $\mathsf{P}(A_n) \to 0$.

a) (8) Does it follow that $\mathsf{E}X_n \mathbf{1}_{A_n} \to 0$? ◯ Yes ◯ No. Prove it, or give a counter-example:

b) (8) Does it follow that $\mathsf{E}X_1 \mathbf{1}_{A_n} \to 0$? ◯ Yes ◯ No. Prove it, or give a counter-example:

c) (4) Would either of your answers to a) or b) change if we have $\{X_n\} \subset L_4(\Omega, \mathcal{F}, \mathsf{P})$ with $\|X_n\|_4 \le 42$? Explain.

Problem 4: Let $\{A_n\}$ be independent events on some probability space $(\Omega, \mathcal{F}, \mathsf{P})$ with $\mathsf{P}(A_n) = 2^{-2n} = 4^{-n}$ and for $n \geq 0$ set

$$X_n := 2^n \mathbf{1}_{A_n} \qquad Y := \sum_{n=0}^{\infty} X_n \qquad Z_n := \prod_{0 \leq j < n} X_j$$

a) (4) Show that Y is finite almost-surely:

b) (4) For which $0 < p \leq \infty$ and $n \in \mathbb{N}$ is $X_n \in L_p$? Why?

c) (4) For which $0 < p \leq \infty$ is $Y \in L_p$? Why?

d) (4) Is the sequence $\{Z_n^p\}$ a Martingale for some $p > 0$? ◯ Yes ◯ No
Why?

e) (4) In what sense(s) does Z_n converge, and to what limit?
◯ $a.s.$ ◯ $pr.$ ◯ L_1 ◯ L_2 ◯ L_∞ Limit:

Problem 5: Let $X \sim \mathsf{Po}(\lambda)$ and $Y \sim \mathsf{Ex}(\theta)$ be independent.

a) (4) Is the distribution $\mu(dz)$ of $Z := X \wedge Y$: ◯ Absolutely Continuous, ◯ Discrete, or ◯ Neither?[1] For each $z \in \mathbb{R}$, find:
$\mathsf{P}[Z = z] = \mu(\{z\}) =$

b) (4) Is the distribution $\nu(ds)$ of $S := X + Y$: ◯ Absolutely Continuous, ◯ Discrete, or ◯ Neither? For each $s \in \mathbb{R}$, find:
$\mathsf{P}[S = s] = \nu(\{s\}) =$

c) (4) For fixed $\theta > 0$, find $\lambda \in \mathbb{R}_+$ to achieve $\mathsf{P}[X < Y] = \frac{1}{2}$:
$\lambda =$

[1] Recall $x \wedge y := \min(x, y)$ for $x, y \in \mathbb{R}$

Problem 5 (cont'd): Still $X \sim \mathsf{Po}(\lambda)$, $Y \sim \mathsf{Ex}(\theta)$, and $X \perp\!\!\!\perp Y$.

d) (4) Find the MGF $M(t) := \mathsf{E}\big[\exp(tS)\big]$ for the sum $S := X + Y$. For which $t \in \mathbb{R}$ is $M(t) < \infty$ finite?
$M(t) =$

e) (4) Find the ch.f. $\chi(\omega) := \mathsf{E}\big[\exp(i\omega S)\big]$ for the sum $S := X + Y$. For which $\omega \in \mathbb{R}$ is $|\chi(\omega)| < \infty$ finite?
$\chi(\omega) =$

Problem 6: The random variables X and Z are independent, with distributions

$$X \sim \mathsf{No}(0,1) \qquad \mathsf{P}[Z=+1] = 1/2 = \mathsf{P}[Z=-1]$$

and product $Y := XZ$. Simplify all answers.

a) (6) What is the probability distribution of Y?

b) (4) What is the covariance of X and Y?

c) (5) Are X and Y independent? ◯ Yes ◯ No Why?

d) (5) Are Y and Z independent? ◯ Yes ◯ No Why?

Problem 7: Let $X \sim \mathsf{Ex}(a)$, $Y \sim \mathsf{Ex}(b)$, and $Z \sim \mathsf{Ex}(c)$ be independent RVs on $(\Omega, \mathcal{F}, \mathsf{P})$ (see *p.* 13 for the pdf, mean, *etc.* of $\mathsf{Ex}(\lambda)$). Find (and simplify):

a) (5) $\mathsf{P}[Y < Z] = \int_0^\infty \int_y^\infty b\, c\, e^{-by-cz}\, dz\, dy =$

b) (5) $\mathsf{P}[X < Y < Z] =$

c) (5) $\mathsf{P}[X < Y \mid Y < Z] =$

d) (5) $\mathsf{E}[XY/Z] =$

Problem 8: True or false? Circle one, for 2 points each. No explanations are needed. All random variables are real on the same space $(\Omega, \mathcal{F}, \mathsf{P})$.

a) T F For any real number $x \in \mathbb{R}$, $x \le e^x - 1$.

b) T F If events A, B, C are independent then $(A \cup B)$ and C are independent too.

c) T F If events A, B, C are independent then $(A \cap B)$ and C are independent too.

d) T F If $\{X_i\}$ are iid and $\mathsf{P}[X_i \in (0,1)] = 1$ then $\prod_{j=1}^n X_j \to 0$ *a.s.*

e) T F If the distribution $\mu(B) := \mathsf{P}[X \in B]$ of X is neither absolutely-continuous nor discrete, then it must be singular continuous.

f) T F If $\mathsf{E}|Z| = \infty$ then $\mathsf{E}[Z \mid \mathcal{G}]$ is not defined.

g) T F If $X \sim \mathsf{Ex}(\lambda)$ then, for $x > 0$, $\mathsf{P}[X = x] = \lambda e^{-\lambda x}$.

h) T F If $\mathsf{E}[e^{i\omega X}] = \cos(\omega)$ then $X \sim \mathsf{Un}\big((-\pi, \pi]\big)$.

i) T F If $\sigma(X) = \sigma(Y)$ then $X = \phi(Y)$ for some Borel function ϕ.

j) T F If $\{M_n\}$ is a martingale then $\xi_n := (M_n - M_{n-1})$ are independent L_1 RVs with mean zero.

Blank Worksheet

Name	Notation	pdf/pmf	Range	Mean μ	Variance σ^2	
Beta	$\mathsf{Be}(\alpha,\beta)$	$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$x\in(0,1)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	
Binomial	$\mathsf{Bi}(n,p)$	$f(x)=\binom{n}{x}p^x q^{(n-x)}$	$x\in 0,\cdots,n$	$n\,p$	$n\,p\,q$	$(q=1-p)$
Exponential	$\mathsf{Ex}(\lambda)$	$f(x)=\lambda\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	$1/\lambda$	$1/\lambda^2$	
Gamma	$\mathsf{Ga}(\alpha,\lambda)$	$f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	α/λ	α/λ^2	
Geometric	$\mathsf{Ge}(p)$	$f(x)=p\,q^x$	$x\in\mathbb{Z}_+$	q/p	q/p^2	$(q=1-p)$
		$f(y)=p\,q^{y-1}$	$y\in\{1,\ldots\}$	$1/p$	q/p^2	$(y=x+1)$
HyperGeo.	$\mathsf{HG}(n,A,B)$	$f(x)=\frac{\binom{A}{x}\binom{B}{n-x}}{\binom{A+B}{n}}$	$x\in 0,\cdots,n$	$n\,P$	$n\,P\,(1-P)\frac{N-n}{N-1}$	$(P=\frac{A}{A+B})$
Logistic	$\mathsf{Lo}(\mu,\beta)$	$f(x)=\frac{e^{-(x-\mu)/\beta}}{\beta[1+e^{-(x-\mu)/\beta}]^2}$	$x\in\mathbb{R}$	μ	$\pi^2\beta^2/3$	
Log Normal	$\mathsf{LN}(\mu,\sigma^2)$	$f(x)=\frac{1}{x\sqrt{2\pi\sigma^2}}e^{-(\log x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}_+$	$e^{\mu+\sigma^2/2}$	$e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right)$	
Neg. Binom.	$\mathsf{NB}(\alpha,p)$	$f(x)=\binom{x+\alpha-1}{x}p^\alpha\,q^x$	$x\in\mathbb{Z}_+$	$\alpha q/p$	$\alpha q/p^2$	$(q=1-p)$
		$f(y)=\binom{y-1}{y-\alpha}p^\alpha\,q^{y-\alpha}$	$y\in\{\alpha,\ldots\}$	α/p	$\alpha q/p^2$	$(y=x+\alpha)$
Normal	$\mathsf{No}(\mu,\sigma^2)$	$f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}$	μ	σ^2	
Pareto	$\mathsf{Pa}(\alpha,\epsilon)$	$f(x)=(\alpha/\epsilon)(1+x/\epsilon)^{-\alpha-1}$	$x\in\mathbb{R}_+$	$\frac{\epsilon}{\alpha-1}$ if $\alpha>1$	$\frac{\epsilon^2\alpha}{(\alpha-1)^2(\alpha-2)}$ if $\alpha>2$	
		$f(y)=\alpha\,\epsilon^\alpha/y^{\alpha+1}$	$y\in(\epsilon,\infty)$	$\frac{\epsilon\,\alpha}{\alpha-1}$ if $\alpha>1$	$\frac{\epsilon^2\alpha}{(\alpha-1)^2(\alpha-2)}$ if $\alpha>2$	$(y=x+\epsilon)$
Poisson	$\mathsf{Po}(\lambda)$	$f(x)=\frac{\lambda^x}{x!}e^{-\lambda}$	$x\in\mathbb{Z}_+$	λ	λ	
Snedecor F	$F(\nu_1,\nu_2)$	$f(x)=\frac{\Gamma(\frac{\nu_1+\nu_2}{2})(\nu_1/\nu_2)^{\nu_1/2}}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})}\times$ $x^{\frac{\nu_1-2}{2}}\left[1+\frac{\nu_1}{\nu_2}x\right]^{-\frac{\nu_1+\nu_2}{2}}$	$x\in\mathbb{R}_+$	$\frac{\nu_2}{\nu_2-2}$ if $\nu_2>2$	$\left(\frac{\nu_2}{\nu_2-2}\right)^2\frac{2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)}$ if $\nu_2>4$	
Student t	$t(\nu)$	$f(x)=\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}}[1+x^2/\nu]^{-(\nu+1)/2}$	$x\in\mathbb{R}$	0 if $\nu>1$	$\frac{\nu}{\nu-2}$ if $\nu>2$	
Uniform	$\mathsf{Un}(a,b)$	$f(x)=\frac{1}{b-a}$	$x\in(a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	
Weibull	$\mathsf{We}(\alpha,\beta)$	$f(x)=\alpha\beta\,x^{\alpha-1}\,e^{-\beta\,x^\alpha}$	$x\in\mathbb{R}_+$	$\frac{\Gamma(1+\alpha^{-1})}{\beta^{1/\alpha}}$	$\frac{\Gamma(1+2/\alpha)-\Gamma^2(1+1/\alpha)}{\beta^{2/\alpha}}$	