

Sta 711: Homework 2

σ -Algebras

1. A *partition* of a measurable space (Ω, \mathcal{F}) is a (finite or countable) collection of disjoint and non-empty events $\Lambda_j \in \mathcal{F}$ with $\Omega = \cup_j \Lambda_j$. Let $\{A, B\} \subset \mathcal{F}$ be two non-empty disjoint events with $A \cup B \neq \Omega$. Enumerate the elements of the partition smallest partition $\mathcal{P} := \mathcal{P}(A, B)$ containing A, B and of the σ -algebra $\sigma(\mathcal{P})$ it generates. How many elements are in $\sigma(\mathcal{P})$?
2. Consider two classes of sets $\mathcal{G} \subset \mathcal{F}$ and $\mathcal{H} \subset \mathcal{F}$ on a measurable space (Ω, \mathcal{F}) . Suppose that for every event $A \in \mathcal{H}$ there exists events $\{B_i\} \subset \mathcal{G}$ such that $A = \cup_i B_i$. Further suppose that for every $B \in \mathcal{G}$ there exists a set $A \in \mathcal{H}$ such that $A^c = B$. Show that the sigma-algebras $\sigma(\mathcal{G})$ and $\sigma(\mathcal{H})$ they generate are equal.

Null sets.

3. Let $\{A_n, n \in \mathbb{N}\}$ be events with $P(A_n) = 1$. Prove that $P(\cap_{n=1}^{\infty} A_n) = 1$.
4. Now consider uncountably many events $\{B_\alpha\}$, all with $P(B_\alpha) = 1$. Does it follow necessarily that $P(\cap_\alpha B_\alpha) = 1$? Give a proof or a counter example.
5. Let $n \in \mathbb{N}$ and let $\{C_k\}$ be a collection of n events such that $\sum_{k=1}^n P(C_k) > n - 1$. Show that $P(\cap_{k=1}^n C_k) > 0$.

Distribution functions and continuity.

6. Give an example¹ of a real-valued function on \mathbb{R} which is continuous, but **not** uniformly continuous.
7. Let G be a continuous distribution function on \mathbb{R} . Show² that G is in fact *uniformly* continuous, *i.e.*, that $(\forall \epsilon > 0)(\exists \delta > 0)(\forall x, y \in \mathbb{R}) |x - y| < \delta \Rightarrow |G(x) - G(y)| < \epsilon$.
8. Show that any distribution function F on \mathbb{R} can have *at most countably many* discontinuities. Hint: Consider the open intervals $(F(x-), F(x))$ for discontinuity points x .
9. Let $\{A_n\}_{n \in \mathbb{N}} \subset \mathcal{F}$ be *increasing* in the sense that each $A_n \subset A_{n+1}$. Prove that $P(A_n) \rightarrow P(\cup_{n \in \mathbb{N}} A_n)$, a property called “continuity”. What happens for $\{B_n\} \subset \mathcal{F}$ with $B_n \supset B_{n+1}$?

¹and, of course, *prove* that your example satisfies the criteria.

²Hint: Consider points $\{x_i\}$ for which $G(x_i) = i/n$ for $1 \leq i < n$. Must these exist? If they do, are they determined uniquely? Does that matter? Draw a graph!

