Sta 711: Homework 3

Random variables

1. Let $(\Omega, \mathcal{F}, \mathsf{P}) = ((0,1], \mathcal{B}, \lambda)$ for Lebesgue measure λ on the Borel sets of the unit interval. For $\omega \in \Omega$ define:

$$X_1(\omega) := \min(\omega, 0.6) \qquad X_2(\omega) := \mathbf{1}_{(0,1/3]}(\omega) \qquad X_3(\omega) := \sqrt{\omega}$$

   Plot each of the CDFs $F_k(x) := \mathsf{P}[X_k \le x]$, $x \in \mathbb{R}$, and describe explicitly the σ-algebras $\mathcal{F}_k := \sigma(X_k)$.

2. Let X be a random variable with CDF $F(x) := \mathsf{P}(X \le x)$. Set $Y := F(X)$. If X has a continuous distribution (*i.e.*, if F is a continuous function), show that Y is a random variable and that Y has a uniform distribution on $[0,1]$. Warning: $F(x)$ may not be *strictly* increasing, and so may not be one-to-one; also it may not be differentiable.

3. A random variable Y is *real-valued* if $Y(\omega) \in \mathbb{R}$ for every $\omega \in \Omega$, and is *bounded* if there is a fixed finite number $0 \le B < \infty$ for which $|Y(\omega)| \le B$ for all $\omega \in \Omega$. Give an example of a real-valued random variable X that is *not* bounded.

4. Let X be a real valued random variable (so $\mathsf{P}[|X| < \infty] = 1$) with CDF $F(x)$. For each $\epsilon > 0$, construct a *bounded* random variable Y_ϵ such that

$$\mathsf{P}(X \neq Y_\epsilon) < \epsilon.$$

Measurable functions

5. Set $\mathcal{S} := \sigma\big(\{(-\infty, 0]\}\big)$ on $\Omega := \mathbb{R}$. Describe all Borel functions $f : \Omega \to \mathbb{R}$ that are $\mathcal{S}\backslash\mathcal{B}$-measurable.

6. If X is a real-valued random variable on any probability space $(\Omega, \mathcal{F}, \mathsf{P})$, then show that $|X|$ is also a random variable. Show by an example that the converse need not be true (Hint: A finite Ω will suffice)

7. Let $(\Omega, \mathcal{F})$ and $(E, \mathcal{E})$ be measurable spaces, and let $X : \Omega \to E$ be any function. Show that $\mathcal{H} := \{B \in \mathcal{E} : \ X^{-1}(B) \in \mathcal{F}\}$ is a σ-algebra.

8. Again let $(\Omega, \mathcal{F})$ and $(E, \mathcal{E})$ be measurable spaces and $X : \Omega \to E$ any function. If $\mathcal{E} = \sigma(\mathcal{C})$ is generated by a class $\mathcal{C} \subset \mathcal{E}$ of sets, show that X is $\mathcal{F}\backslash\mathcal{E}$-measurable if and only $X^{-1}(C) \in \mathcal{F}$ for each $C \in \mathcal{C}$.

9. Let $\mathcal{F}_X := \sigma(X)$ be the σ-algebra generated by the function $X(\omega) := \omega^2$ on $\Omega = \mathbb{R}$. Is the set $A = (-\infty, 0]$ in $\mathcal{F}_X$? How about $B = [-4, 4]$? Why?

10. Let $\{X_n, n \geq 0\}$ be real-valued random variables on $(\Omega, \mathcal{F}, \mathsf{P})$ that satisfy

$$\limsup_{n\to\infty} X_n(\omega) = +\infty$$

for every $\omega \in \Omega$, and let $B < \infty$ be a real number. Prove that the integer-valued quantity

$$\tau(\omega) := \inf\{n \geq 0 : X_n(\omega) \geq B\}$$

is a random variable.

Extra credit: Prove that X_τ is also a random variable.

Random Variables and σ-Algebras

11. All parts of this problem concern the same probability space $(\Omega, \mathcal{F}, \mathsf{P})$ with $\Omega = (0, 1]$, $\mathcal{F} = \mathcal{B}(\Omega)$ the Borel sets, and $\mathsf{P} = \lambda$ Lebesgue measure. Let $\delta_n(\omega)$ be the nth bit in the binary expansion of ω, given by

$$\delta_n(\omega) := \lceil 1 + 2^n \omega \rceil \pmod 2$$

where $\lceil x \rceil$ is the least integer $\geq x$, and set

$$\mathcal{F}_n := \sigma\{\delta_1, \ldots, \delta_n\} = \sigma\{(0, j/2^n] : \; j = 0, \cdots, 2^n\}.$$

(a) Find a single real-valued random variable X on $(\Omega, \mathcal{F}, \mathsf{P})$ such that $\mathcal{F}_3 = \sigma(X)$.

(b) True or False: If Y is any other random variable on $(\Omega, \mathcal{F}, \mathsf{P})$ such that $\mathcal{F}_3 = \sigma(Y)$, then $Y = g(X)$ for some Borel measurable function $g : \mathbb{R} \to \mathbb{R}$. Give a proof (find g explicitly) or a counter-example.

(c) Let Z be a random variable on $(\Omega, \mathcal{F}, \mathsf{P})$ for which $\mathcal{F} = \sigma(Z)$ (recall $\mathcal{F} = \mathcal{B}(\Omega)$, $\Omega = (0, 1]$, and $\mathsf{P} = \lambda$). True or false: For each $\omega_1 \neq \omega_2$, necessarily $Z(\omega_1) \neq Z(\omega_2)$. Explain.