

Sta 711: Homework 4

Expectation

Feel free to use the result of one problem in your solution to a subsequent problem.

1. Let $X := (X_1, X_2)$ be distributed uniformly over the triangle in \mathbb{R}^2 with vertices $\{(-1, 0), (1, 0), (0, 1)\}$. Evaluate $\mathbf{E}(X_1 + X_2)$ (no need to approximate by simple functions, just find the value and show how you did it).
2. Let $X \geq 0$ be a nonnegative random variable on $(\Omega, \mathcal{F}, \mathbf{P})$ and, for $n \in \mathbb{N}$, set

$$X_n(\omega) := \min(2^n, 2^{-n} \lfloor 2^n X(\omega) \rfloor).$$

Prove that X_n is simple (how many distinct values can it take on?) and $X_n \nearrow X$. Note you must show *both* monotonicity and convergence. For $\omega \in \Omega$ and $\epsilon > 0$, how big must n be to ensure $|X - X_n| < \epsilon$?

3. Suppose $X \in L_1(\Omega, \mathcal{F}, \mathbf{P})$, i.e., $\mathbf{E}|X| < \infty$. Show that¹

$$\int_{|X|>m} X d\mathbf{P} \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

4. Let $\{A_n\}$ denote a sequence of events such that $\mathbf{P}(A_n) \rightarrow 0$ as $n \rightarrow \infty$ and let $X \in L_1$. Show that

$$\mathbf{E}[X \mathbf{1}_{A_n}] = \int_{A_n} X d\mathbf{P} \rightarrow 0$$

Hint: Use problem 3. Warning: these conditions do *not* imply that $X \mathbf{1}_{A_n} \rightarrow 0$ *a.s.*

5. Fix a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and define a distance measure d on \mathcal{F} by $d(A, B) := \mathbf{P}(A \Delta B)$ where (as usual) $A \Delta B := (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$ denotes the symmetric difference. Show that, if $\{A_n\} \subset \mathcal{F}$ and $A \in \mathcal{F}$ satisfy $d(A_n, A) \rightarrow 0$, then

$$\int_{A_n} X d\mathbf{P} \rightarrow \int_A X d\mathbf{P}$$

for every $X \in L_1(\Omega, \mathcal{F}, \mathbf{P})$. Hint: Use problem 4.

¹The “expectation of a random variable X over an event A ” can be written in many ways, including $\int_A X d\mathbf{P} = \mathbf{E}[X \mathbf{1}_A] = \int_A X(\omega) \mathbf{P}(d\omega)$.

Convergence Theorems

6. Let $X \geq 0$ be a non-negative random variable. Define sequences of random variables X_n and of extended real numbers $0 \leq S_n \leq \infty$ for positive integers $n \in \mathbb{N}$ by:

$$X_n := \sum_{k=0}^{\infty} \frac{k}{2^n} \mathbf{1}_{\{k < 2^n X \leq k+1\}} \quad S_n := \mathbf{E}X_n = \sum_{k=0}^{\infty} \frac{k}{2^n} \mathbf{P} \left\{ \frac{k}{2^n} < X \leq \frac{k+1}{2^n} \right\}$$

Is X_n “simple”? What is $\lim_{n \rightarrow \infty} S_n$? Justify your answers.

7. Define a sequence of random variables on $(\Omega, \mathcal{F}, \mathbf{P}) = ((0, 1], \mathcal{B}, \lambda)$ by

$$X_n := \frac{n}{\log(n+1)} \mathbf{1}_{(0, \frac{1}{n}]} \quad n \in \mathbb{N}.$$

Show that $\mathbf{P}[X_n \rightarrow 0] = 1$, and that $\mathbf{E}(X_n) \rightarrow 0$. Also show that the Dominated Convergence Theorem does not apply to this example. Why?

8. Let $\{Y_n\}$ be a sequence of random variables for $n \in \mathbb{N}$ with

$$\mathbf{P}(Y_n = -n^3) = \mathbf{P}(Y_n = +n^3) = \frac{1}{2n^2}, \quad \mathbf{P}(Y_n = 0) = 1 - \frac{1}{n^2}$$

One can (but you don't have to) use the Borel-Cantelli lemma to show that $Y_n \rightarrow 0$ *a.s.* Compute $\lim_{n \rightarrow \infty} \mathbf{E}(Y_n)$ and $\lim_{n \rightarrow \infty} \mathbf{E}(|Y_n|)$. Is the Dominated Convergence Theorem applicable? Why or why not?

9. Let $\{X_n\}, X$ be random variables with $0 \leq X_n \rightarrow X$. If $\sup_n \mathbf{E}(X_n) \leq K < \infty$, show that $X \in L_1$ and $\mathbf{E}(X) \leq K$. Does $X_n \rightarrow X$ in L_1 ?

Domination

10. Let $\{X_n\}$ be a sequence of random variables. We have seen that $\{X_n\}$ is dominated by some integrable Y , *i.e.*, $|X_n| \leq Y$ and $\mathbf{E}Y < \infty$, if and only if

$$\mathbf{E} \left(\sup_{n \in \mathbb{N}} |X_n| \right) < \infty \tag{1}$$

Thus, (1) is exactly equivalent to domination in Lebesgue's sense (but Lebesgue's domination is often easier to verify). Does the condition

$$\sup_{n \in \mathbb{N}} \mathbf{E}(|X_n|) < \infty \tag{2}$$

imply (1)? Or is it implied *by* (1)? For each direction ($1 \Rightarrow 2$ and $2 \Rightarrow 1$), give either a proof or a counter-example.