Sta 711: Homework 5

Convergence

1. Let X be a strictly positive random variable. Show that:

   (a) $\lim_{n\to\infty} n\mathsf{E}(\frac{1}{X}\mathbf{1}_{[X>n]}) = 0$.

   (b) $\lim_{n\to\infty} n^{-1}\mathsf{E}(\frac{1}{X}\mathbf{1}_{[X>n^{-1}]}) = 0$.

2. Let $X \sim \mathsf{Un}(0,4]$ be uniformly distributed on the interval $(0,4]$, and set $Y := 1/X$ and $Z := \log(4Y)$. Suggestion: First find out what is the distribution of Z, by computing $\mathsf{P}[Z > z]$ for $z \in \mathbb{R}$. Use $\varphi(x) := |x|$ for the Markov inequality questions.

   (a) What bound does Markov's inequality give for $\mathsf{P}[X > 3]$?

   (b) What bound does Chebychev's inequality give for $\mathsf{P}\big[|X - 2| > 1\big]$?

   (c) What bound does Markov's inequality give for $\mathsf{P}[Y > 1]$?

   (d) What bound does Markov's inequality give for $\mathsf{P}[Z > 2]$?

   (e) What are the exact values of $\mathsf{P}[X > 3]$, $\mathsf{P}\big[|X - 2| > 1\big]$, $\mathsf{P}[Y > 1]$, and $\mathsf{P}[Z > 2]$?

3. Let A and B be events in $(\Omega, \mathcal{F}, \mathsf{P})$ with probabilities $a = \mathsf{P}(A)$ and $b = \mathsf{P}(B)$ respectively. Show that $\mathsf{P}(A \cap B) \le \sqrt{ab}$.

4. Suppose $\{X_n\}, X$ are real valued RVs defined on a probability space $(\Omega, \mathcal{F}, \mathsf{P})$ and that $X_n(\omega) \to X(\omega)$ for all $\omega \in \Omega$. Show that for every $\epsilon > 0$, there is an event Λ_ϵ with $\mathsf{P}(\Lambda_\epsilon) < \epsilon$ and

   $$\sup_{\omega \in \Lambda_\epsilon^c} |X(\omega) - X_n(\omega)| \to 0 \qquad \text{as } n \to \infty.$$

   Thus the convergence is uniform except on an arbitrarily small set. (For more on this result, called Egorov's Theorem, see page 89 of the text.)

5. For a random variable X, $1 < p < q < \infty$, show[1] that

   $$0 \le \|X\|_1 \le \|X\|_p \le \|X\|_q \le \|X\|_\infty$$

6. For $1 < p < q < \infty$, show that

   $$L_\infty \subset L_q \subset L_p \subset L_1$$

   where $L_p := \{X : \ \|X\|_p < \infty\}$.

[1] Hint: Jensen's inequality may help for some parts

7. The "Moment Generating Function" (MGF) of a real-valued random variable X (or of its distribution $\mu(dx)$) is the extended real-valued function $M_X(t) := \mathsf{E}\exp(tX) = \int_{\mathbb{R}} e^{tx}\,\mu(dx)$ of $t \in \mathbb{R}$. Show that a nonnegative random variable $X \geq 0$ is in L_1 if $M_X(t) < \infty$ for any $t > 0$. Show that the converse may fail— *i.e.*, there exist $X \geq 0$ in L_1 for which $M_X(t) = \infty$ for all $t > 0$.

8. Show that Minkowski's Inequality fails for $0 < p < 1$— *i.e.*, find $(\Omega, \mathcal{F}, \mathsf{P})$ and $X, Y \in L_p(\Omega, \mathcal{F}, \mathsf{P})$ for which $\|X + Y\|_p > \|X\|_p + \|Y\|_p$ for some $0 < p < 1$.