Sta 711: Homework 9

Uniform Integrability

1. **True or false?** Answer whether each of the following statements is true or false. If true, answer why; if false, give a simple counter example.

   (a) If $\{X_n,\ n \in \mathbb{N}\}$ is a uniformly integrable (UI) collection of random variables, then X_n is uniformly bounded in L_1.

   (b) Define a sequence $\{X_n\}$ of random variables on the unit interval with Lebesgue measure, $(\Omega, \mathcal{F}, P)$ with $\Omega = (0, 1]$, $\mathcal{F} = \mathcal{B}$, and $\mathsf{P} = \lambda$, by $X_n := \sqrt{n}\mathbf{1}_{(0,\frac{1}{n}]}$. Then $\{X_n\}$ is UI.

   (c) Let $\{X_n\}$ be a sequence of random variables for which $e^{|X_n|}$ is uniformly bounded in L_1, *i.e.*, satisfies $\mathsf{E}e^{|X_n|} \le B$ for some $B < \infty$ and all n. Then $\{X_n\}$ is UI.

   (d) Let $\{X_n\}$ be a sequence of random variables that is uniformly bounded in L_1, *i.e.*, satisfies $\mathsf{E}|X_n| \le B$ for some $B < \infty$ and all n. Then $\{X_n\}$ is UI.

Characteristic Functions

2. Let X be a random variable, and define

$$\phi_X(\theta) := \mathsf{E}(e^{i\theta X}), \qquad \theta \in \mathbb{R}$$

   Show that $\phi_X(\theta)$ is uniformly continuous in $\mathbb{R}$.

3. Find the characteristic functions of the following random variables:

   (a) $W := c$[1] (The superscripts in (a)–(c) are footnote indicators, not exponents)

   (b) $X \sim \mathsf{Un}(a, b)$[2]

   (c) $Y \sim \mathsf{Ga}(\alpha, \lambda)$[3]

   (d) $Z_n = (Y_1 + Y_2 + \cdots + Y_n)/n, \quad Y_j \overset{\text{iid}}{\sim} \mathsf{Ga}(\alpha, \lambda)$

   What is the distribution of Z_n? What happens as $n \to \infty$?

4. The distribution of a random variable X is called *infinitely divisible* if, for every $n \in \mathbb{N}$, there exist n iid random variables $\{Y_i\}$ such that X has the same distribution as $\sum_{i=1}^n Y_i$. Use characteristic functions to show that if $X \sim \mathsf{Po}(\lambda)$, then X is infinitely divisible.[4]

[1] A constant random variable with value $c \in \mathbb{R}$

[2] Uniform, on the interval $(a, b) \subset \mathbb{R}$

[3] Gamma, with rate parameterization— with pdf $f(y \mid \alpha, \lambda) = \lambda^\alpha y^{\alpha-1} e^{-\lambda y}/\Gamma(\alpha)$, $y > 0$.

[4] Hint: If $\{Y_i\}$ are independent with sum $Y_+ := \sum Y_i$, then $\phi_{Y_+}(\theta) = \prod \phi_{Y_i}(\theta)$ for all $\theta \in \mathbb{R}$.

5. Show that the ch.f. $\phi(\theta) := \mathsf{E}[\exp(i\theta X)]$ of an RV X is real-valued if and only if X and $Y := -X$ have the same distribution, if and only if $\phi(\theta) = \phi(-\theta)$ for each $\theta \in \mathbb{R}$.

6. Let $X, Y \overset{\text{iid}}{\sim} \mathsf{Ex}(1)$. Find the pdf $f(x)$ and ch.f. $\phi(\theta)$ for $Z := X - Y$. Simplify! Note, from your answer to 5) above, $i = \sqrt{-1}$ should not appear in your answer.

7. Let X, Y be iid with mean zero and variance one, with symmetric distributions (so X, $-X$, Y, and $-Y$ all have the same distribution). If $(X + Y)$ and $(X - Y)$ are independent, show that $X, Y \overset{\text{iid}}{\sim} \mathsf{No}(0, 1)$. Hint: if X_1 and $(X_1 + X_2 + X_3 + X_4)/2$ have the same distribution, with $\{X_j\}$ iid and L_2, what does CLT say?

8. Let $\{X_j\} \overset{\text{iid}}{\sim} \mathsf{Po}(\lambda)$, $S_n := \sum_{j=1}^n X_j$, and $Z_n := S_n/\sqrt{n} - \lambda\sqrt{n}$.

   - Find the ch.f. $\phi_1(\theta) = \mathsf{E}e^{i\theta X_j}$ for X_j.
   - Find the ch.f. $\phi_n(\theta) = \mathsf{E}e^{i\theta S_n}$ for S_n.
   - Find the ch.f. $\chi_n(\theta) = \mathsf{E}e^{i\theta Z_n}$ for Z_n.
   - Find the limit $\lim_{n\to\infty} \chi_n(\theta)$.
     What does this say about the approximate distribution for Z_n, for large n?