Basics of Probability

STA 102: Introduction to Biostatistics

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The following material was used by Yue Jiang during a live lecture.

Without the accompanying oral comments, the text is incomplete as a record of the presentation.

What's the use of probability?

- Last time: how descriptive statistics are used to *describe* data
- Goal: Make *inferences* about a population based on a sample

To do this, we need a solid foundation of probability theory.

Probabilities come up all the time

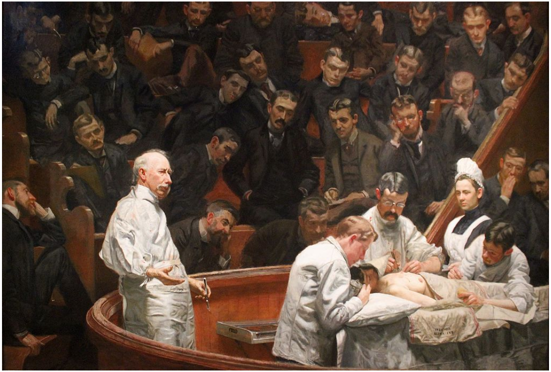
How do we interpret the following statements?

- There is a moderate chance of drought in North Carolina during the next year
- The surgery has a 50-50 probability of success
- The ten-year survival probability of invasive breast cancer among U.S women is 83%

Interpretations of probability

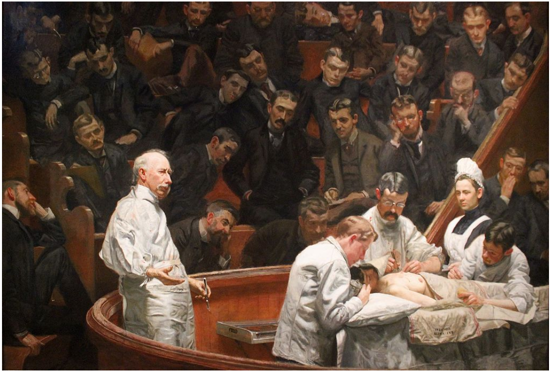
"There is a 1 in 3 chance of selecting a white ball"

Interpretations of probability



"The surgery has a 50% probability of success"

Interpretations of probability



Long-run frequencies vs. degree of belief

Probability spaces

Mathematical objects that model random experiments.

A probability space consists of three components:

1. A sample space, the set of all possible outcomes
2. Subsets of the sample space, called events, which comprise any number of possible outcomes (including none of them!)
3. A function that assigns probabilities to events

An event occurs if the outcome of the random experiment is contained in that event

Sample spaces

Sample spaces depend on the random experiment in question

- Tossing a single fair coin
- Tossing two fair coins
- Sum of rolling two fair six-sided dice
- Survival (years) after cancer diagnosis

Events

Subsets of the sample space that comprise possible outcomes. Essentially, these are all the 'plausibly reasonable' events we're interested in calculating probabilities for*:

- Tossing a single fair coin
- Tossing two fair coins
- Sum of rolling two fair six-sided dice
- Survival (years) after cancer diagnosis*

**there are some nasty mathematical details behind this seemingly simple task. Don't worry about them!*

Probabilities

A number describing the likelihood of each event's occurrence.
This maps events to a number between 0 and 1, inclusive:

- Tossing a single fair coin A head
- Tossing two fair coins At least one head
- Sum of rolling two fair six-sided dice An odd number
- Survival (years) after cancer diagnosis $>$one year

Probabilities

A number describing the likelihood of each event's occurrence. This maps events to a number between 0 and 1, inclusive:

- Tossing a single fair coin A head 0.5
- Tossing two fair coins At least one head 0.75
- Sum of rolling two fair six-sided dice An odd number 0.5
- Survival (years) after cancer diagnosis >one year ...harder

Events as (sub)sets

Let's take for now the example of tossing a single fair coin and recording the outcome.

There are only two elements in the outcome space:

- A: getting a head
- B: getting a tail

We can define the simple events of A occurring or B occurring, but are there "other" events we can define?

Set operations

For two sets (or events) A and B, the most common relationships are:

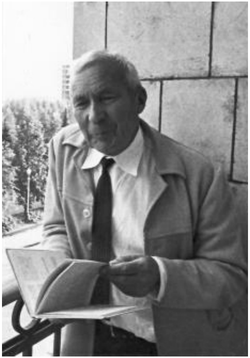
- Intersection ($A \cap B$): A and B both occur
- Union ($A \cup B$): A or B occur (including when both occur)
- Complement (A^c): A does not occur
- Difference ($A \setminus B$): A occurs, but B does not occur: $A \cap B^c$

Two sets A and B are said to be disjoint if $A \cap B = \emptyset$

How do probabilities "work"?

Kolmogorov axioms

1. The probability of any event in the sample space is a non-negative real number
2. The probability of the entire sample space is 1
3. If A and B are disjoint events (mutually exclusive), then the probability of A or B occurring is the sum of the individual probabilities that they occur



How do probabilities "work"?

For two events A and B with probabilities $P(A)$ and $P(B)$ of occurring, the Kolmogorov axioms give us two important rules:

- Complement Rule: $P(A^c) = 1 - P(A)$
- Inclusion-Exclusion: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

How do we extend inclusion-exclusion to more than two events?

DeMorgan's laws

- Complement of union:
  $(A \cup B)^c = A^c \cap B^c$
- Complement of intersection:
  $(A \cap B)^c = A^c \cup B^c$

How do we extend DeMorgan's laws to more than two events?

Conditional probability

The probability an event will occur when another event has already occurred. The conditional probability of event A given event B is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Examples come up all the time in the real-world:

- *Given* that a mammogram comes back positive, what is the probability that a woman has breast cancer?
- *Given* that a 68-year old man has suffered four previous heart attacks, what is the probability he die in the next five years?
- *Given* that a patient has a mutation in the *CFTR* gene, what is the probability their offspring will have cystic fibrosis?

Gunter et al. (2017) study

ORIGINAL RESEARCH

Annals of Internal Medicine

Coffee Drinking and Mortality in 10 European Countries

A Multinational Cohort Study

	Died		
Coffee drinking	Yes	No	Total
None	1039	5438	6477
Med-Low	4440	29712	34152
High	3601	24934	28535
Total	9080	60084	69164

Define the events $A =$ died and $B =$ non-coffee drinker

- Marginal probability $P(A)$
- Joint probability $P(A \cap B)$
- Conditional probability $P(A|B)$

More practice

Coffee drinking	Died Yes	No	Total
None	1039	5438	6477
Med-Low	4440	29712	29809
High	3601	24934	28535
Total	9080	60084	64821

- ...did not drink coffee?
- ...died during the study or did not drink coffee?
- ...did not die during the study and was a high coffee drinker?
- ...died during the study given that they do not drink coffee?