Bayes' Rule

STA 102: Introduction to Biostatistics

Yue Jiang

Review: conditional probability

The probability an event will occur when another event has already occurred. The conditional probability of event A given event B is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Examples come up all the time in the real-world:

- *Given* that a mammogram comes back positive, what is the probability that a woman has breast cancer?
- *Given* that a 68-year old man has suffered four previous heart attacks, what is the probability he die in the next five years?
- *Given* that a patient has a mutation in the *CFTR* gene, what is the probability their offspring will have cystic fibrosis?

Independence and the multiplicative rule

We can rewrite the definition of conditional probability:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \implies \underbrace{P(A \cap B) = P(A|B) \times P(B)}_{\text{Multiplicative Rule}}$$

What does the multiplicative rule mean in plain English?

Events A and B are said to be independent when

$$P(A \cap B) = P(A) \times P(B)$$

or equivalently, $P(A|B) = P(A)$ or $P(B|A) = P(B)$

Independent vs. disjoint events

Since for two independent events $P(A|B) = P(A)$ or $P(B|A) = P(B)$, knowing that one event has occurred tells us nothing more about the probability of the other occurring.

For two disjoint events A and B, knowing that one has occurred tells us that the other definitely has not occurred: $P(A \cap B) = 0$

Conditional probabilities and independence

Original Research — Annals of Internal Medicine

Coffee Drinking and Mortality in 10 European Countries
A Multinational Cohort Study

What was the probability a randomly selected person in the study...

	Died		
Coffee drinking	Yes	No	Total
None	1039	5438	6477
Med-Low	4440	29712	34152
High	3601	24934	28535
Total	9080	60084	69164

- ...died?
- ...died, given they were a non-coffee drinker?

In this study, were dying and coffee drinking independent events? How might we check?

The law of total probability

Suppose we partition B into mutually disjoint events $B_1, B_2, \cdots B_k$ that comprise the entirety of *the entire sample space*.

The law of total probability states that the probability of event A is

$$\begin{aligned} P(A) &= P(A \cap B_1) + P(A \cap B_2) + \cdots + P(A \cap B_k) \\ &= P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + \cdots + P(A|B_k)P(B_k) \end{aligned}$$

The law of total probability in action

ORIGINAL RESEARCH — Annals of Internal Medicine

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	Died		
Coffee drinking	Yes	No	Total
None	1039	5438	6477
Med-Low	4440	29712	34152
High	3601	24934	28535

What was the probability a randomly selected person died?

The law of total probability in action

In an introductory statistics course, 50% of students were first years, 30% were sophomores, and 20% were upperclassmen.

80% of the first years didn't get enough sleep, 40% of the sophomores didn't get enough sleep, and 10% of the upperclassmen didn't get enough sleep.

What is the probability that a randomly selected student in this class didn't get enough sleep?

Bayes' rule

What is the probability that a random person...

Coffee drinking	Died: Yes	Died: No	Total
None	1039	5438	6477
Med-Low	4440	29712	34152
High	3601	24934	28535
Total	9080	60084	69164

- ...was a high coffee drinker, given that they died?
- ...died, given that they were a high coffee drinker?

These are $P(High|Died)$ and $P(Died|High)$.

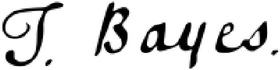
Are these two probabilities the same?

Bayes' rule

(A real portrait of him doesn't seem to exist)

We can use Bayes' rule to "reverse" the order of conditioning. By definition:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B|A)P(A)}{P(B)}$$



Bayes' rule

Using the definition of conditional probability, the law of total probability, and the multiplicative rule, we have

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B|A)P(A)}{P(B)}$$

$$= \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)}$$

If instead A is partitioned into k mutually disjoint events that together comprise the entire sample space, then

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

$$= \frac{P(B|A)P(A)}{P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + \cdots + P(B|A_k)P(A_k)}$$

Bayes' rule

What is the probability that a random person in the EPIC study...

	Died		
Coffee drinking	Yes	No	Total
None	1039	5438	6477
Med-Low	4440	29712	34152
High	3601	24934	28535
Total	9080	60084	69164

- ...was a high coffee drinker, given that he died?
- ...died, given that he was a high coffee drinker?

Verify these results using Bayes' rule

Looking ahead