Final Examination

STA 711: Probability & Measure Theory

Saturday, 2017 Dec 16, 7:00 – 10:00 pm

This is a closed-book exam. You may use a sheet of prepared notes, if you wish, but you may not share materials.

If a question seems ambiguous or confusing, *please* ask me to clarify it. Unless a problem states otherwise, you must **show** your **work**. There are blank worksheets at the end of the test if you need more room for this, and also a pdf/pmf sheet.

It is to your advantage to write your solutions as clearly as possible, and to box answers I might not find.

For full credit, answers must be given in **closed form** with no unevaluated sums, integrals, maxima, unreduced fractions. Wherever, possible **simplify**.

Good luck.

1.	/20	5.	/20
2.	/20	6.	/20
3.	/20	7.	/20
4.	/20	8.	/20
	/80		/80
Total:	/160		

Print Name: ______________________________

Problem 1: Let $\xi_1, \xi_2, \ldots$ be iid random variables with the $\mathsf{Ex}(1/2)$ distribution (hence mean $\mathsf{E}[\xi_j] = 2$... see distribution reference sheet, *p.* 15).

a) (8) Find non-random $a_n \in \mathbb{R}$, $b_n > 0$ such that $S_n := \sum_{1 \le j \le n} \xi_j$ satisfies

$$\mathsf{P}[(S_n - a_n)/b_n \le x] \to F(x)$$

for a non-trivial df F (*i.e.*, one for a distribution not concentrated at a single point). Give a_n, b_n, and F. Justify your answer.

Problem 1 (cont'd): Still $\{\xi_j\} \overset{\text{iid}}{\sim} \mathsf{Ex}(1/2)$.

b) (6) Find non-random $a_n \in \mathbb{R}$, $b_n > 0$ such that $X_n := \min_{1\le j\le n} \xi_j$ satisfies:

$$\mathsf{P}[(X_n - a_n)/b_n \le x] \to G(x)$$

for a non-trivial df G. Give a_n, b_n, and G. Justify your answer.

c) (6) Find non-random $a_n \in \mathbb{R}$, $b_n > 0$ such that $Y_n := \max_{1\le j\le n} \xi_j$ satisfies

$$\mathsf{P}[(Y_n - a_n)/b_n \le x] \to H(x)$$

for a non-trivial df H. Give a_n, b_n, and H. Justify your answer.

Problem 2: Let $\{X_n\}$ and Y be real-valued random variables on $(\Omega, \mathcal{F}, \mathsf{P})$ such that $X_n \to Y$ (pr).

a) (10) Set $A_n := \{\omega : |X_n(\omega)| > n\}$. Prove that $\mathsf{P}(A_n) \to 0$ as $n \to \infty$.

b) (10) Prove that $\exp(-X_n^2) \to \exp(-Y^2)$ in $L_1(\Omega, \mathcal{F}, \mathsf{P})$.

Problem 3: For each part below, select “True” or “False” and sketch a **short explanation** or counter-example to support your answer:

a) (4) T F If $\{X_j\}$ are L_1 random variables and $\sum \|X_j\|_1 < \infty$ then $S_n := \sum_{1\le j\le n} X_j$ converges in L_1 to a limit $S \in L_1(\Omega, \mathcal{F}, \mathsf{P})$.

b) (4) T F If $\{X_n\}$, Y are L_2 random variables with $\|X_n\|_2 \le 10$ and if $X_n \to Y$ in L_1 then $\mathsf{P}[X_n \to Y] = 1$.

c) (4) T F If $\{X_n\}$, Y are L_1 random variables with $X_1 \le 42$ *a.s.* and if $X_n \searrow Y$ decreases to Y *a.s.*, then $X_n \to Y$ in L_1.

d) (4) T F If $\{X_j\}$ are independent L_1 random variables with zero mean $\mathsf{E}[X_j] = 0$ then $Y_n := 1 + \sum_{1\le j\le n} j^2 X_j$ is a martingale.

e) (4) T F If $X \in L_p$ for every $0 < p < \infty$ then also $X \in L_\infty$, because $\|X\|_p \to \|X\|_\infty$ as $p \to \infty$.

Problem 4: Let $\Omega = \mathbb{N} = \{1, 2, \dots\}$ be the natural numbers, with probability measure

$$\mathsf{P}(A) := \frac{6}{\pi^2} \sum_{\omega \in A} \frac{1}{\omega^2}$$

on the power set $A \in \mathcal{F} := 2^\Omega$. Note $\mathsf{P}(\Omega) = 1$ because $\sum_{\omega=1}^{\infty} \frac{1}{\omega^2} = \pi^2/6$.

a) (4) Let $E := \{2j : j \in \mathbb{N}\}$ be the even numbers, $D := \{2^j : j \in \mathbb{N}\}$ the integer powers of two that are ≥ 2, and $S := \{j^2 : j \in \mathbb{N}\}$ the squares. How many events are in each of the following classes? (*Events* $A \subset \Omega$, not *elements* $\omega \in A$)

$\sigma(E, S)$: ____________ $\sigma(D, E)$: ____________

$\pi(D, E)$: ____________ $\sigma(D, E, S)$: ____________

b) (8) Find the indicated probabilities:

$\mathsf{P}(E) =$ $\mathsf{P}(D) =$

Problem 4 (cont'd): Still $\Omega = \mathbb{N}$ and $\mathsf{P}(A) := \frac{6}{\pi^2} \sum_{\omega \in A} \frac{1}{\omega^2}$.

c) (8) Set $X(\omega) = \mathbf{1}_{\{\omega \le 3\}}$. Find:

$$\mathsf{E}(X \mid \sigma(E)) =$$

Problem 5: Let $X_j \overset{\text{iid}}{\sim} \mathsf{Po}(1)$ be independent random variables, all with the unit-mean Poisson distribution.

a) (8) Find the logarithm of the ch.f. of X_j, $\phi(\omega) := \mathsf{E}\big[e^{i\omega X_j}\big]$:
$\psi(\omega) = \log\phi(\omega) =$

b) (6) For numbers $a > 0$, find the log ch.f. $\psi_1(\omega)$ of $(X_j - 1)/a$.
$\psi_1(\omega) =$

c) (6) Let $S_n = X_1 + \cdots + X_n$ be the partial sum. Find a sequence $a_n > 0$ such that the log characteristic function $\psi_n(\omega)$ of $(S_n - n)/a_n$ converges to $-\omega^2/2$ for every ω, and explain what this says about the limiting probability distribution of S_n (*i.e.*, about the $\mathsf{Po}(n)$ distribution for large n).[1]

[1] Recall the Taylor series $e^x = 1 + x + x^2/2 + o(x^2) \approx 1 + x + x^2/2$ near $x \approx 0$.

Problem 6: Miscellaneous examples & counter-examples. Let $\{X_n\}$, X, and Y be real-valued RVs on a space $(\Omega, \mathcal{F}, \mathsf{P})$, and let $\mu(dx)$ and $\nu(dy)$ be the probability distribution measures of X and Y, respectively.

a) (5) Suppose X and Y are independent, and $g : \mathbb{R}^2 \to \mathbb{R}$ is a Borel function. Sometimes it's okay to switch orders of integration to evaluate the expectation $\mathsf{E}[g(X,Y)] = \iint g(x,y)\,(\mu \otimes \nu)(dx\,dy)$ as either of:

$$\int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} g(x,y)\mu(dx) \right\} \nu(dy) \stackrel{?}{=} \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} g(x,y)\nu(dy) \right\} \mu(dx)$$

and sometimes it's not. What are the two different sets of broadly-applicable conditions on g, μ, ν given by Fubini's Theorem , either of which will ensure equality of these two expressions?

1.

2.

b) (5) Even if $\{X_n\}$ and X are in L_1, and $X_n \to X$ in probability, it's possible that $\mathsf{E}X_n$ does not converge to $\mathsf{E}X$ and that $\mathsf{E}|X_n - X|$ does not converge to zero. Give an example of $\{X_n\}$ and X in L_1 where $X_n \to X$ *(pr.)* but L_1 convergence fails.

Problem 6 (cont'd): More miscellaneous examples & counter-examples.

c) (5) Give an example of an RV X on $(\Omega, \mathcal{F}, \mathsf{P})$ with $\Omega = (0, 1]$, $\mathcal{F} = \mathcal{B}(\Omega)$, and Lebesgue P that is in L_1 but not in L_2.
$X(\omega) =$

d) (5) Give an example of a Martingale $(X_n, \mathcal{F}_n)$ with filtration $\mathcal{F}_n = \sigma\{X_j : \ 0 \le j \le n\}$ and a (finite) stopping time τ for which $\mathsf{E}[X_0] \ne \mathsf{E}[X_\tau]$.

Problem 7: More miscellany.

a) (10) The standard Cauchy distribution $\mathsf{Ca}(0,1)$ has pdf

$$f(x) = \frac{1/\pi}{1+x^2}, \qquad x \in \mathbb{R}$$

and famously has no mean, with $\mathsf{E}\big[|X|\big] = \infty$ for $X \sim \mathsf{Ca}(0,1)$. For any $0 \le p < 1$, however, $\|X\|_p^p = \mathsf{E}\big[|X|^p\big] < \infty$. Find and prove[2] a (numerical) finite upper bound

$\mathsf{E}\big[|X|^{1/2}\big] \le$ ____________________

[2]Suggestion: First use symmetry to focus on $\mathbb{R}_+$; then worry separately about $[0,1]$ and $(1,\infty)$.

Problem 7 (cont'd): Yet more miscellany. Will it never end?

b) (5) Let X, Y be RVs on $(\Omega, \mathcal{F}, \mathsf{P})$, with $X \in L_4$ and $Y \in L_p$. For which $p > 0$ is $XY \in L_1$? Why?

c) (5) If sequences $\{X_n\}$ and $\{Y_n\}$ of RVs on $(\Omega, \mathcal{F}, \mathsf{P})$ satisfy

$$\mathsf{P}[X_n > Y_n] \le 2^{-n}$$

for each $n \in \mathbb{N}$, does it follow that $\limsup X_n \le \liminf Y_n$ almost surely? Give a proof or counter-example.

Problem 8: Circle True or False; no explanations are needed.

a) T F If $X_n \to X$ $(pr.)$ then $\limsup_{n\to\infty} X_n = X$.

b) T F If X on $(\Omega, \mathcal{F}, \mathsf{P})$ has a cont. dist'n then Ω is uncountable.

c) T F If $g(\cdot)$ is a bounded Borel function on $\mathbb{R}$ and $X_n \to X$ $(pr.)$ then $g(X_n) \to g(X)$ $(pr.)$.

d) T F If $0 < X < \infty$ and $\mathsf{E}[1/X] = 1/\mathsf{E}[X]$ then $X \in L_\infty$.

e) T F If $X \perp\!\!\!\perp Y$ and $\mathsf{P}[X < Y] = \mathsf{P}[X > Y] = 1/2$ then X, Y have the same distribution.

f) T F If $X \perp\!\!\!\perp Z$ and $Y \perp\!\!\!\perp Z$ then $(X + Y) \perp\!\!\!\perp Z$.

g) T F If $X \perp\!\!\!\perp Z$ and $Y = e^X$ then $Y \perp\!\!\!\perp Z$.

h) T F If probability measures P, Q agree on a λ-system $\mathcal{L}$ then they agree on the π-system $\mathcal{P} = \pi(\mathcal{L})$ it generates.

i) T F If $X^* := \limsup_{n\to\infty} X_n$ is a non-constant random variable, then $\{X_n\}$ cannot be independent.

j) T F If X has a discrete dist'n and Y has a continuous one, then $(X + Y)$ must have a continuous distribution (even if X, Y are not independent).

Blank Worksheet

Another Blank Worksheet

Name	Notation	pdf/pmf	Range	Mean μ	Variance σ^2	
Beta	$\mathsf{Be}(\alpha,\beta)$	$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$x\in(0,1)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	
Binomial	$\mathsf{Bi}(n,p)$	$f(x)=\binom{n}{x}p^x q^{(n-x)}$	$x\in 0,\cdots,n$	$n\,p$	$n\,p\,q$	$(q=1-p)$
Exponential	$\mathsf{Ex}(\lambda)$	$f(x)=\lambda\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	$1/\lambda$	$1/\lambda^2$	
Gamma	$\mathsf{Ga}(\alpha,\lambda)$	$f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	α/λ	α/λ^2	
Geometric	$\mathsf{Ge}(p)$	$f(x)=p\,q^x$	$x\in\mathbb{Z}_+$	q/p	q/p^2	$(q=1-p)$
		$f(y)=p\,q^{y-1}$	$y\in\{1,...\}$	$1/p$	q/p^2	$(y=x+1)$
HyperGeo.	$\mathsf{HG}(n,A,B)$	$f(x)=\frac{\binom{A}{x}\binom{B}{n-x}}{\binom{A+B}{n}}$	$x\in 0,\cdots,n$	$n\,P$	$n\,P\,(1-P)\frac{N-n}{N-1}$	$(P=\frac{A}{A+B})$
Logistic	$\mathsf{Lo}(\mu,\beta)$	$f(x)=\frac{e^{-(x-\mu)/\beta}}{\beta[1+e^{-(x-\mu)/\beta}]^2}$	$x\in\mathbb{R}$	μ	$\pi^2\beta^2/3$	
Log Normal	$\mathsf{LN}(\mu,\sigma^2)$	$f(x)=\frac{1}{x\sqrt{2\pi\sigma^2}}e^{-(\log x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}_+$	$e^{\mu+\sigma^2/2}$	$e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right)$	
Neg. Binom.	$\mathsf{NB}(\alpha,p)$	$f(x)=\binom{x+\alpha-1}{x}p^\alpha\,q^x$	$x\in\mathbb{Z}_+$	$\alpha q/p$	$\alpha q/p^2$	$(q=1-p)$
		$f(y)=\binom{y-1}{y-\alpha}p^\alpha\,q^{y-\alpha}$	$y\in\{\alpha,...\}$	α/p	$\alpha q/p^2$	$(y=x+\alpha)$
Normal	$\mathsf{No}(\mu,\sigma^2)$	$f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}$	μ	σ^2	
Pareto	$\mathsf{Pa}(\alpha,\epsilon)$	$f(x)=(\alpha/\epsilon)(1+x/\epsilon)^{-\alpha-1}$	$x\in\mathbb{R}_+$	$\frac{\epsilon}{\alpha-1}$ if $\alpha>1$	$\frac{\epsilon^2\alpha}{(\alpha-1)^2(\alpha-2)}$ if $\alpha>2$	
		$f(y)=\alpha\,\epsilon^\alpha/y^{\alpha+1}$	$y\in(\epsilon,\infty)$	$\frac{\epsilon\,\alpha}{\alpha-1}$ if $\alpha>1$	$\frac{\epsilon^2\alpha}{(\alpha-1)^2(\alpha-2)}$ if $\alpha>2$	$(y=x+\epsilon)$
Poisson	$\mathsf{Po}(\lambda)$	$f(x)=\frac{\lambda^x}{x!}e^{-\lambda}$	$x\in\mathbb{Z}_+$	λ	λ	
Snedecor F	$F(\nu_1,\nu_2)$	$f(x)=\frac{\Gamma(\frac{\nu_1+\nu_2}{2})(\nu_1/\nu_2)^{\nu_1/2}}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})}\times$ $x^{\frac{\nu_1-2}{2}}\left[1+\frac{\nu_1}{\nu_2}x\right]^{-\frac{\nu_1+\nu_2}{2}}$	$x\in\mathbb{R}_+$	$\frac{\nu_2}{\nu_2-2}$ if $\nu_2>2$	$\left(\frac{\nu_2}{\nu_2-2}\right)^2\frac{2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)}$ if $\nu_2>4$	
Student t	$t(\nu)$	$f(x)=\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}}[1+x^2/\nu]^{-(\nu+1)/2}$	$x\in\mathbb{R}$	0 if $\nu>1$	$\frac{\nu}{\nu-2}$ if $\nu>2$	
Uniform	$\mathsf{Un}(a,b)$	$f(x)=\frac{1}{b-a}$	$x\in(a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	
Weibull	$\mathsf{We}(\alpha,\beta)$	$f(x)=\alpha\beta\,x^{\alpha-1}\,e^{-\beta\,x^\alpha}$	$x\in\mathbb{R}_+$	$\frac{\Gamma(1+\alpha^{-1})}{\beta^{1/\alpha}}$	$\frac{\Gamma(1+2/\alpha)-\Gamma^2(1+1/\alpha)}{\beta^{2/\alpha}}$	