Midterm Examination II

STA 711: Probability & Measure Theory

Wednesday, 2017 Nov 15, 1:25 – 2:40pm

This is a closed-book exam. You may use a single sheet of prepared notes, if you wish, but you may not share materials.

If a question seems ambiguous or confusing, *please* ask me to clarify it. Unless a problem states otherwise, you must **show** your **work**. There are blank worksheets at the end of the test if you need more room for this, and also a pdf/pmf sheet.

It is to your advantage to write your solutions as clearly as possible, and to box answers I might not find.

For full credit, answers must be given in **closed form** with no unevaluated sums, integrals, maxima, unreduced fractions. Wherever possible, **Simplify**.

Good luck!

Print Name Clearly: ______________________

1.	/20
2.	/20
3.	/20
4.	/20
5.	/20
Total:	/100

Problem 1: Let X and Y be independent, each with mean $\mathsf{E}X = \mathsf{E}Y = 2$, but not identically distributed— X has a Geometric distribution[1] with pmf $\mathsf{P}[X = x] = p\,(1-p)^x$ for $x \in \mathbb{Z}_0 = \{0, 1, \dots\}$ for some $0<p<1$, and Y has an Exponential distribution with pdf $\lambda e^{-\lambda y}\mathbf{1}_{\{y>0\}}$ for some $\lambda > 0$. Find the indicated quantities (as **numeric values**). Show your work.

a) (4) $\mathsf{P}[X \geq 1] =$ $\qquad$ $\mathsf{P}[Y \geq 1] =$

b) (4) $\mathsf{P}[X = 1] =$ $\qquad$ $\mathsf{P}[Y = 1] =$

c) (4) $\mathsf{P}[Y \geq X] =$ $\qquad$ $\mathsf{V}[X - Y] =$

[1] Common distributions' pdfs/pmfs, means, variances, *etc.* are attached as page 10.

Problem 1 (cont'd): Still $X \perp\!\!\!\perp Y$ and $\mathsf{E}X = \mathsf{E}Y = 2$, with $X \sim \mathsf{Ge}(p)$ and $Y \sim \mathsf{Ex}(\lambda)$ for some $p \in (0,1)$ and $\lambda > 0$:

d) (4) $\mathsf{E}\exp(i\omega X) =$ $\qquad$ $\mathsf{E}\exp(i\omega Y) =$ $\qquad$ $(\omega \in \mathbb{R})$

e) (4) $\mathsf{E}(1/X!) =$ $\qquad$ $\mathsf{E}Y^5 =$

Problem 2: Let $\{X_n\} \overset{\text{iid}}{\sim} \mathsf{Ex}(1)$ be iid unit-rate exponential random vara-iables on some space $(\Omega, \mathcal{F}, \mathsf{P})$. In each part below, indicate in which (if any) sense(s) the sequence $\{Y_n\}$ converges to zero. No explanations are necessary.

a) (5) $Y_n := X_n/n$: ○ $a.s.$ ○ $pr.$ ○ L_1 ○ L_2 ○ L_∞

b) (5) $Y_n := \big\{\prod_{1\le j\le n} X_j\big\}^{1/n}$: ○ $a.s.$ ○ $pr.$ ○ L_1 ○ L_2 ○ L_∞

c) (5) $Y_n := \frac{1}{n}\sum_{1\le j\le n}(X_j - 1)$: ○ $a.s.$ ○ $pr.$ ○ L_1 ○ L_2 ○ L_∞

d) (5) $Y_n := \min_{1\le j\le n} X_j$: ○ $a.s.$ ○ $pr.$ ○ L_1 ○ L_2 ○ L_∞

e) (XC) Prove that $Y_n := \big\{\prod_{1\le j\le n} X_j\big\} \to 0$ $a.s.$ but not in L_1.

Problem 3: Let $\{X_n\} \overset{\text{ind}}{\sim} \mathsf{Pa}(n, 1)$ be independent Pareto random variables with $\mathsf{P}[X_n > x] = x^{-n}$ for $x > 1$ and $n \in \mathbb{N}$. Show your work in finding:

a) (4) For every $0 < p \leq \infty$ and $n \in \mathbb{N}$, find:
$\|X_n\|_p =$

b) (4) For $n \neq m$, find:
$\mathsf{P}[X_m > X_n] =$

c) (4) Does X_n converge almost-surely? Prove your answer. ◯ Yes ◯ No
Why?

Problem 3 (cont'd): Still $\{X_n\} \overset{\text{ind}}{\sim} \mathsf{Pa}(n,1)$ w/ $\mathsf{P}[X_n > x] = x^{-n}$ for $x > 1$.

d) (4) Set $T_n := \prod_{j=1}^n X_j$ and $Z_n := \prod_{j=1}^n X_{j^2}$. Show that $T_n \to \infty$ *a.s.* but $Z_n \to Z$ *pr.* for some finite RV Z.

e) (4) Set $Y_n := (X_n - 1)/X_n$. Does $\sum_{n=1}^\infty Y_n$ converge in L_1? ◯ Yes ◯ No
If so, prove it; if not, find a subsequence n_k s.t. $\sum_{k=1}^\infty Y_{n_k}$ converges.

Problem 4: If X, Y, and Z are i.i.d. L_1 with common mean μ, ch.f. $\phi(\omega)$, and sums $S := X + Y + Z$ and $T := Y + Z$, find:

a) (4) $\mathsf{E}[S \mid Y] =$

b) (4) $\mathsf{E}[Y \mid S] =$

c) (4) $\mathsf{E}[X \mid Y] =$

d) (4) $\mathsf{E}[X + Y \mid T] =$

e) (4) $\mathsf{E}[e^{i\omega S} \mid Y] =$

Problem 5: True or false? Circle one; each answer is worth 2 points. No explanations are needed, but you can give one if you think a question seems ambiguous or tricky. All random variables are real on some $(\Omega, \mathcal{F}, \mathsf{P})$.

a) T F For the Cauchy distribution, $\mathsf{E}[\exp(tX)]$ is infinite for all $t \in \mathbb{R}$ except for $t = 0$ because the Cauchy pdf has heavy tails.

b) T F If $\{X_i\}$ are iid w/ch.f. $\phi(\omega)$, then $-\sum_{j=1}^n X_j$ has ch.f. $\phi(-\omega)^n$.

c) T F If $\{X, Y, Z\}$ are iid and $\mathsf{P}[X < Y < Z] = 1/6$ then X has a continuous distribution.

d) T F If X and Y are independent with pdfs $f(x)$ and $g(y)$, then $Z := X \cdot Y$ has pdf $h(z) := f(z)\, g(z)$.

e) T F If $\{X_n\}$ are iid and L_∞ with mean $\mu = \mathsf{E}X_n$ then $(\forall \epsilon > 0)\ (\exists c_\epsilon > 0)\ (\forall n \in \mathbb{N})\ \mathsf{P}[(\bar{X}_n - \mu) > \epsilon] \le \exp(-n\, c_\epsilon)$.

f) T F If $\mathsf{E}|X_n|^4 \to 0$ then also $\mathsf{E}|X_n|^{\frac{1}{4}} \to 0$.

g) T F If $\mathcal{G} \subset \mathcal{F}$ and $Y = \mathsf{E}[X \mid \mathcal{G}]$ with $0 \le X \in L_1$, then $\|X\|_1 = \|Y\|_1$.

h) T F If $\mathcal{G} \subset \mathcal{F}$ and $Y = \mathsf{E}[X \mid \mathcal{G}]$ with $0 \le X \in L_2$, then $\|X\|_2 = \|Y\|_2$.

i) T F Every ch.f. $\phi(\omega) = \mathsf{E}\left[e^{i\omega X}\right]$ is a continuous function of ω.

j) T F If $X_n \to X$ in L_1 then, for some $n_k \to \infty$, $X_{n_k} \to X$ *a.s.*

Blank Worksheet

Another Blank Worksheet

Name	Notation	pdf/pmf	Range	Mean μ	Variance σ^2	
Beta	Be(α, β)	$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$	$x \in (0,1)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	
Binomial	Bi(n, p)	$f(x) = \binom{n}{x} p^x q^{(n-x)}$	$x \in 0, \cdots, n$	$n\,p$	$n\,p\,q$	$(q = 1-p)$
Exponential	Ex(λ)	$f(x) = \lambda\, e^{-\lambda x}$	$x \in \mathbb{R}_+$	$1/\lambda$	$1/\lambda^2$	
Gamma	Ga(α, λ)	$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1}\, e^{-\lambda x}$	$x \in \mathbb{R}_+$	α/λ	α/λ^2	
Geometric	Ge(p)	$f(x) = p\, q^x$	$x \in \mathbb{Z}_+$	q/p	q/p^2	$(q = 1-p)$
		$f(y) = p\, q^{y-1}$	$y \in \{1, ...\}$	$1/p$	q/p^2	$(y = x+1)$
HyperGeo.	HG(n, A, B)	$f(x) = \frac{\binom{A}{x}\binom{B}{n-x}}{\binom{A+B}{n}}$	$x \in 0, \cdots, n$	$n\,P$	$n\,P\,(1-P)\frac{N-n}{N-1}$	$(P = \frac{A}{A+B})$
Logistic	Lo(μ, β)	$f(x) = \frac{e^{-(x-\mu)/\beta}}{\beta[1+e^{-(x-\mu)/\beta}]^2}$	$x \in \mathbb{R}$	μ	$\pi^2\beta^2/3$	
Log Normal	LN(μ, σ^2)	$f(x) = \frac{1}{x\sqrt{2\pi\sigma^2}} e^{-(\log x - \mu)^2/2\sigma^2}$	$x \in \mathbb{R}_+$	$e^{\mu+\sigma^2/2}$	$e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right)$	
Neg. Binom.	NB(α, p)	$f(x) = \binom{x+\alpha-1}{x} p^\alpha\, q^x$	$x \in \mathbb{Z}_+$	$\alpha q/p$	$\alpha q/p^2$	$(q = 1-p)$
		$f(y) = \binom{y-1}{y-\alpha} p^\alpha\, q^{y-\alpha}$	$y \in \{\alpha, ...\}$	α/p	$\alpha q/p^2$	$(y = x+\alpha)$
Normal	No(μ, σ^2)	$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$	$x \in \mathbb{R}$	μ	σ^2	
Pareto	Pa(α, ϵ)	$f(x) = (\alpha/\epsilon)(1+x/\epsilon)^{-\alpha-1}$	$x \in \mathbb{R}_+$	$\frac{\epsilon}{\alpha-1}^*$	$\frac{\epsilon^2\alpha}{(\alpha-1)^2(\alpha-2)}^*$	
		$f(y) = \alpha\, \epsilon^\alpha / y^{\alpha+1}$	$y \in (\epsilon, \infty)$	$\frac{\epsilon\,\alpha}{\alpha-1}^*$	$\frac{\epsilon^2\alpha}{(\alpha-1)^2(\alpha-2)}^*$	$(y = x+\epsilon)$
Poisson	Po(λ)	$f(x) = \frac{\lambda^x}{x!} e^{-\lambda}$	$x \in \mathbb{Z}_+$	λ	λ	
Snedecor F	$F(\nu_1, \nu_2)$	$f(x) = \frac{\Gamma(\frac{\nu_1+\nu_2}{2})(\nu_1/\nu_2)^{\nu_1/2}}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} \times x^{\frac{\nu_1-2}{2}} \left[1 + \frac{\nu_1}{\nu_2} x\right]^{-\frac{\nu_1+\nu_2}{2}}$	$x \in \mathbb{R}_+$	$\frac{\nu_2}{\nu_2-2}^*$	$\left(\frac{\nu_2}{\nu_2-2}\right)^2 \frac{2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)}^*$	
Student t	$t(\nu)$	$f(x) = \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}}[1+x^2/\nu]^{-(\nu+1)/2}$	$x \in \mathbb{R}$	0^*	$\nu/(\nu-2)^*$	
Uniform	Un(a, b)	$f(x) = \frac{1}{b-a}$	$x \in (a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	
Weibull	We(α, β)	$f(x) = \alpha\beta\, x^{\alpha-1}\, e^{-\beta\, x^\alpha}$	$x \in \mathbb{R}_+$	$\frac{\Gamma(1+\alpha^{-1})}{\beta^{1/\alpha}}$	$\frac{\Gamma(1+2/\alpha)-\Gamma^2(1+1/\alpha)}{\beta^{2/\alpha}}$	