# Midterm Examination II

STA 711: Probability & Measure Theory

Wednesday, 2018 Nov 14, 1:25 – 2:40pm

This is a closed-book exam. You may use a single sheet of prepared notes, if you wish, but you may not share materials.

If a question seems ambiguous or confusing, *please* ask me to clarify it. Unless a problem states otherwise, you must **show** your **work**. There are blank worksheets at the end of the test if you need more room for this, and also a pdf/pmf sheet.

It is to your advantage to write your solutions as clearly as possible, and to box answers I might not find.

For full credit, answers must be given in **closed form** with no unevaluated sums, integrals, maxima, unreduced fractions. Wherever possible, **Simplify**.

Good luck!

Print Name Clearly: ______________________________

| | |
|---|---|
| 1. | /20 |
| 2. | /20 |
| 3. | /20 |
| 4. | /20 |
| 5. | /20 |
| Total: | /100 |

**Problem 1**: Let $\Omega := (0, \infty)$ with the Borel sets for $\mathcal{F}$ and probability measure given by

$$\mathsf{P}\{(a, b]\} = e^{-a} - e^{-b}, \qquad 0 < a < b < \infty.$$

Let $X_n(\omega) := \omega^n/n!$ and $Y_n := X_n/2^n$ on $\Omega$ for $n \in \mathbb{N}_0 = \{0, 1, 2, \cdots\}$, with partial sums $S_n := \sum_{j=0}^n X_j$ and $T_n := \sum_{j=0}^n Y_j$.

a) (2) Find the limits. Simplify!:

$S := \lim_{n\to\infty} S_n =$ $\qquad\qquad$ $T := \lim_{n\to\infty} T_n =$

b) (2) Show that $\mathsf{E}[X_n] = 1$ for all $n \in \mathbb{N}_0$.

c) (4) For all $n \in \mathbb{N}_0$ and $p \geq 1$, find

$\|X_n\|_p =$ ____________________

**Problem 1 (cont'd)**: Still $\mathsf{P}(d\omega) = e^{-\omega}\,d\omega$ on $\Omega = \mathbb{R}_+$, $X_n(\omega) := \omega^n/n!$, $Y_n := X_n/2^n$, $S_n := \sum_{j=0}^{n} X_j$ and $T_n := \sum_{j=0}^{n} Y_j$.

d) (4) Does the Monotone Convergence Theorem apply to $\{S_n\}$? ◯ Yes ◯ No. If so, what does it say? If not, why not?

e) (4) Does the Dominated Convergence Theorem apply to $\{S_n\}$? ◯ Yes ◯ No. If so, what does it say? What's the dominator? If not, why not?

f) (4) Does the Dominated Convergence Theorem apply to $\{T_n\}$? ◯ Yes ◯ No. If so, what does it say? What's the dominator? If not, why not?

**Problem 2**: Let $\{X_n\}$ be iid with CDF

$$F(x) = \mathsf{P}[X_n \le x] = 1 - (1+x)^{-2}, \qquad x > 0$$

and set $S_n := \sum_{j=1}^{n} X_j$ and $M_n := \min_{1\le j\le n} X_j$.

a) (2) Find the probability density function $f(x)$ for $X_n$.

b) (4) Fix $n \in \mathbb{N}$. For which $p > 0$ is $X_n \in L_p$?

c) (5) Does $S_n/n$ converge as $n \to \infty$? ◯ Yes ◯ No
If so, to what limit, in what sense[1], and why? If not, why not?

d) (5) Fix $n \in \mathbb{N}$. For what $p > 0$ is $M_n \in L_p$?

e) (4) Does $M_n$ converge as $n \to \infty$? ◯ Yes ◯ No
If so, to what limit, in what sense[1], and why? If not, why not?

[1] In case it converges in more than one sense, give any correct sense with a matching answer to "why?".

**Problem 3**: Still $\{X_n\}$ are IID with CDF $F(x) = 1-(1+x)^{-2}$ for $x > 0$.

a) (4) Does the Central Limit Theorem apply to $\{X_n\}$? ◯ Yes ◯ No
If so, what does it say? If not, why not?

b) (4) Let $Y_n := \mathbf{1}_{\{X_n>1\}}$ and $T_n := \sum_{j=1}^{n} Y_j$.
Find the mean and variance of $Y_n$ and $T_n$:

$\mathsf{E}[Y_n]$ = ______________ $\mathsf{E}[T_n]$ = ______________

$\mathsf{V}[Y_n]$ = ______________ $\mathsf{V}[T_n]$ = ______________

**Problem 3 (cont'd)**: Still $\{X_n\}$ are IID with CDF $F(x) = 1-(1+x)^{-2}$ for $x > 0$ and $T_n := \sum_{j=1}^n Y_j$ with $Y_j := \mathbf{1}_{\{X_j>1\}}$.

c) (6) Find the ch.f. $\phi_n(\omega) := \mathsf{E}\exp(i\omega Z_n)$ for $Z_n := (T_n - n/4)/\sqrt{n}$:
$\phi_n(\omega) =$

d) (6) For large $n$ this *has to* be approximately $\phi_n(\omega) \approx \exp(-\kappa\omega^2/2)$ for some $\kappa > 0$. What theorem says so? And what is $\kappa$? $\kappa =$

**Problem 4**: On $(\Omega, \mathcal{F}, \mathsf{P}) = \big((0,1], \mathcal{B}, \lambda\big)$, let $\mathcal{F}_n = \sigma\{(0, j/2^n] : 1 \le j \le 2^n\}$ and let $X(\omega) := 1/\omega$, $Y(\omega) := \omega^2$, $Z(\omega) := \mathbf{1}_{\{(0,3/8]\}}$.

a) (4) Is $\mathsf{E}[X \mid \mathcal{F}_2]$ well-defined? ◯ Yes ◯ No
If so, give its value at $\omega = 1/3$; if not, say why.
$\mathsf{E}[X \mid \mathcal{F}_2](1/3) =$

b) (4) Is $\mathsf{E}[Y \mid \mathcal{F}_2]$ well-defined? ◯ Yes ◯ No
If so, give its value at $\omega = 1/3$; if not, say why.
$\mathsf{E}[Y \mid \mathcal{F}_2](1/3) =$

c) (4) Is $\mathsf{E}[Z \mid \mathcal{F}_2]$ well-defined? ◯ Yes ◯ No
If so, give its value at $\omega = 1/3$; if not, say why.
$\mathsf{E}[Z \mid \mathcal{F}_2](1/3) =$

d) (4) Find the indicated conditional expectation:
$\mathsf{E}[Y \mid Z] =$

e) (4) Find the indicated conditional expectation:
$\mathsf{E}[Z \mid Y] =$

**Problem 5**: True or false? Circle one, for 2 points each. No explanations are needed. All random variables are real on the same space $(\Omega, \mathcal{F}, \mathsf{P})$.

a) T F If RVs $X_n$ *decrease* to a limit $X \in L_1$, and if each $X_n \le Z$ for some $Z \in L_1$, then $X_n \to X$ in $L_1$.

b) T F If $\{X_j\}$ are independent w/ch.f.s $\phi_j(\omega) := \mathsf{E}\big[e^{i\omega X_j}\big]$, then $\sum_{j=1}^n X_j$ has ch.f. $\phi(\omega) := \prod_{j=1}^n \phi_j(\omega)$.

c) T F If $\{X, Y, Z\}$ are iid and $\mathsf{P}[X > 0] = 1$ then $\mathsf{E}[X/(Y+Z)] = 1/2$.

d) T F If $X$ and $Y$ are independent and $Y$ has pdf $f(y)$ then $Z := X + Y$ has an absolutely-continuous distribution too with some pdf $g(z)$.

e) T F If $\{X_n\}$ satisfy $\mathsf{P}[|X_n| \le n] = 1$ and converge *a.s.* to a limit $X$, then $X_n \to X$ in $L_1$.

f) T F If $\mathsf{E}\sqrt{|X_n|} \to 0$ then also $\mathsf{E}|X_n|^2 \to 0$.

g) T F If $\mathsf{P}[A] \le 1/4$ then $\|X\mathbf{1}_A\|_1 \le \frac{1}{2}\|X\|_2$ for any RV $X$.

h) T F If $\mathcal{G} \subset \mathcal{F}$, $Y := \mathsf{E}[X \mid \mathcal{G}]$, and $0 \le X \in L_2$, then $\|X\|_2 \ge \|Y\|_2$.

i) T F If $X$ has ch.f. $\phi(\omega)$ then $Y := \sqrt{X}$ has ch.f. $\phi(\omega/2)$.

j) T F If $X_n \to X$ in $L_2$ then, for some $n_k \to \infty$, $X_{n_k} \to X$ in $L_4$.

**Blank Worksheet**

**Another Blank Worksheet**

| Name | Notation | pdf/pmf | Range | Mean $\mu$ | Variance $\sigma^2$ | |
|---|---|---|---|---|---|---|
| **Beta** | $\mathsf{Be}(\alpha,\beta)$ | $f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$ | $x\in(0,1)$ | $\frac{\alpha}{\alpha+\beta}$ | $\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$ | |
| **Binomial** | $\mathsf{Bi}(n,p)$ | $f(x)=\binom{n}{x}p^x q^{(n-x)}$ | $x\in 0,\cdots,n$ | $n\,p$ | $n\,p\,q$ | $(q=1-p)$ |
| **Exponential** | $\mathsf{Ex}(\lambda)$ | $f(x)=\lambda\,e^{-\lambda x}$ | $x\in\mathbb{R}_+$ | $1/\lambda$ | $1/\lambda^2$ | |
| **Gamma** | $\mathsf{Ga}(\alpha,\lambda)$ | $f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\,e^{-\lambda x}$ | $x\in\mathbb{R}_+$ | $\alpha/\lambda$ | $\alpha/\lambda^2$ | |
| **Geometric** | $\mathsf{Ge}(p)$ | $f(x)=p\,q^x$ | $x\in\mathbb{Z}_+$ | $q/p$ | $q/p^2$ | $(q=1-p)$ |
| | | $f(y)=p\,q^{y-1}$ | $y\in\{1,...\}$ | $1/p$ | $q/p^2$ | $(y=x+1)$ |
| **HyperGeo.** | $\mathsf{HG}(n,A,B)$ | $f(x)=\frac{\binom{A}{x}\binom{B}{n-x}}{\binom{A+B}{n}}$ | $x\in 0,\cdots,n$ | $n\,P$ | $n\,P\,(1-P)\frac{N-n}{N-1}$ | $(P=\frac{A}{A+B})$ |
| **Logistic** | $\mathsf{Lo}(\mu,\beta)$ | $f(x)=\frac{e^{-(x-\mu)/\beta}}{\beta[1+e^{-(x-\mu)/\beta}]^2}$ | $x\in\mathbb{R}$ | $\mu$ | $\pi^2\beta^2/3$ | |
| **Log Normal** | $\mathsf{LN}(\mu,\sigma^2)$ | $f(x)=\frac{1}{x\sqrt{2\pi\sigma^2}}e^{-(\log x-\mu)^2/2\sigma^2}$ | $x\in\mathbb{R}_+$ | $e^{\mu+\sigma^2/2}$ | $e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right)$ | |
| **Neg. Binom.** | $\mathsf{NB}(\alpha,p)$ | $f(x)=\binom{x+\alpha-1}{x}p^\alpha\,q^x$ | $x\in\mathbb{Z}_+$ | $\alpha q/p$ | $\alpha q/p^2$ | $(q=1-p)$ |
| | | $f(y)=\binom{y-1}{y-\alpha}p^\alpha\,q^{y-\alpha}$ | $y\in\{\alpha,...\}$ | $\alpha/p$ | $\alpha q/p^2$ | $(y=x+\alpha)$ |
| **Normal** | $\mathsf{No}(\mu,\sigma^2)$ | $f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-\mu)^2/2\sigma^2}$ | $x\in\mathbb{R}$ | $\mu$ | $\sigma^2$ | |
| **Pareto** | $\mathsf{Pa}(\alpha,\epsilon)$ | $f(x)=(\alpha/\epsilon)(1+x/\epsilon)^{-\alpha-1}$ | $x\in\mathbb{R}_+$ | $\frac{\epsilon}{\alpha-1}$ if $\alpha>1$ | $\frac{\epsilon^2\alpha}{(\alpha-1)^2(\alpha-2)}$ if $\alpha>2$ | |
| | | $f(y)=\alpha\,\epsilon^\alpha/y^{\alpha+1}$ | $y\in(\epsilon,\infty)$ | $\frac{\epsilon\,\alpha}{\alpha-1}$ if $\alpha>1$ | $\frac{\epsilon^2\alpha}{(\alpha-1)^2(\alpha-2)}$ if $\alpha>2$ | $(y=x+\epsilon)$ |
| **Poisson** | $\mathsf{Po}(\lambda)$ | $f(x)=\frac{\lambda^x}{x!}e^{-\lambda}$ | $x\in\mathbb{Z}_+$ | $\lambda$ | $\lambda$ | |
| **Snedecor** $F$ | $F(\nu_1,\nu_2)$ | $f(x)=\frac{\Gamma(\frac{\nu_1+\nu_2}{2})(\nu_1/\nu_2)^{\nu_1/2}}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})}\times x^{\frac{\nu_1-2}{2}}\left[1+\frac{\nu_1}{\nu_2}x\right]^{-\frac{\nu_1+\nu_2}{2}}$ | $x\in\mathbb{R}_+$ | $\frac{\nu_2}{\nu_2-2}$ if $\nu_2>2$ | $\left(\frac{\nu_2}{\nu_2-2}\right)^2\frac{2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)}$ if $\nu_2>4$ | |
| **Student** $t$ | $t(\nu)$ | $f(x)=\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}}[1+x^2/\nu]^{-(\nu+1)/2}$ | $x\in\mathbb{R}$ | $0$ if $\nu>1$ | $\frac{\nu}{\nu-2}$ if $\nu>2$ | |
| **Uniform** | $\mathsf{Un}(a,b)$ | $f(x)=\frac{1}{b-a}$ | $x\in(a,b)$ | $\frac{a+b}{2}$ | $\frac{(b-a)^2}{12}$ | |
| **Weibull** | $\mathsf{We}(\alpha,\beta)$ | $f(x)=\alpha\beta\,x^{\alpha-1}\,e^{-\beta\,x^\alpha}$ | $x\in\mathbb{R}_+$ | $\frac{\Gamma(1+\alpha^{-1})}{\beta^{1/\alpha}}$ | $\frac{\Gamma(1+2/\alpha)-\Gamma^2(1+1/\alpha)}{\beta^{2/\alpha}}$ | |