Midterm Examination I

STA 711: Probability & Measure Theory

Thursday, 2019 Oct 03, 1:25 – 2:40pm

This is a closed-book exam. You may use a single sheet of prepared notes, if you wish, but you may not share materials.

If a question seems ambiguous or confusing, *please* ask me to clarify it. Unless a problem states otherwise, you must **show** your **work**. There are blank worksheets at the end of the test if you need more room for this, and also a pdf/pmf sheet.

It is to your advantage to write your solutions as clearly as possible, and to $\boxed{\text{box}}$ answers I might not find.

For full credit, answers must be given in **closed form** with no unevaluated sums, integrals, limits, maxima, minima, *etc.*, or unreduced fractions. Wherever possible, **Simplify**.

Good luck!

Print Name Clearly: ______________________________

1.	/20
2.	/20
3.	/20
4.	/20
5.	/20
Total:	/100

Version a

Problem 1: Let $X \sim \mathsf{Un}(0,1)$ be a random variable with the standard Uniform distribution.

a) (5) Find a real-valued function $f(x)$ so that $Y := f(X)$ has a non-degenerate[1] distribution with expectation $\mathsf{E}[Y] = 10$, if possible; if not possible, explain why.
$f(x) =$

b) (5) Find a real-valued function $f(x)$ so that $Y := f(X)$ has infinite expectation $\mathsf{E}[Y] = +\infty$, if possible; if not possible, explain why.
$f(x) =$

c) (5) Find a real-valued function $f(x)$ so that $Y := f(X)$ does not have an expectation, if possible; if not possible, explain why.
$f(x) =$

d) (5) Find real-valued functions $f(x)$, $g(x)$ so that $Y := f(X)$ and $Z := g(X)$ each have non-degenerate distributions, and are independent, if possible; if not possible, explain why:
$f(x) =$ $g(x) =$

[1] The distribution of Y is *degenerate* if $\mathsf{P}[Y = c] = 1$ for some constant c; otherwise it is *non-degenerate*. Equivalently, it is degenerate if its DF takes only the values 0 and 1.

Problem 2: Let $\Omega = (0,1]$, $\mathcal{F} = \mathcal{B}(\Omega)$, and $\mathsf{P} = \lambda$ (Lebesgue measure), and fix $\{a_n\} \subset \mathbb{R}$ and $\{b_n\} \subset (0,1]$. Define random variables

$$X_n(\omega) := a_n \mathbf{1}_{\{\omega \le b_n\}}$$

a) (10) For $b_n := 1/n$, what conditions must a_n satisfy to ensure that $\|X_n\|_2 \le 3$ for all n?

b) (10) For $b_n := 1/n^4$ and $a_n = n^c$, for what values of $c \in \mathbb{R}$ does $\sum_{n=1}^{N} X_n$ converge in L_2 as $N \to \infty$?

Problem 3: Let X and Y be RVs in $L_4(\Omega, \mathcal{F}, \mathsf{P})$. Give answers below in terms of $x := \|X\|_4 < \infty$ and $y := \|Y\|_4 < \infty$.

a) (6) Prove that the maximum absolute value $Z := \big(|X| \vee |Y|\big)$ is in L_4 and give an upper bound for $\|Z\|_4$ (in terms of x and y).

b) (8) For which $r > 0$ is $XY \in L_r$? Give an upper bound for $\|XY\|_r$.

c) (6) Let $A \in \mathcal{F}$ be an event with probability $a := \mathsf{P}[A]$. Give and justify a non-trivial upper bound for $\mathsf{E}|X|\mathbf{1}_A = \|X\mathbf{1}_A\|_1$. Recall $x := \|X\|_4$.

Problem 4: Let $(\Omega, \mathcal{F}, \mathsf{P})$ be the unit interval $\Omega = (0, 1]$ with Borel sets $\mathcal{F} = \mathcal{B}$ and Lebesgue measure $\mathsf{P} = \lambda$. Where possible below, evaluate limits numerically.

a) (4) Let $X(\omega) := \omega^{-1/2}$ and $X_n := \min(n, X)$. Find: $\mathsf{E}[X] =$ ________
Does Lebesgue's monotone convergence theorem apply?[2] If so, what does it say? If not, why? ◯ Yes ◯ No Reasoning:

b) (4) Again $X(\omega) := \omega^{-1/2}$ and $X_n := \min(n, X)$. Does Lebesgue's dominated convergence theorem apply? If so, what is a dominating RV $Y \in L_1(\Omega, \mathcal{F}, \mathsf{P})$? If not, why? ◯ Yes ◯ No Reasoning:

[2]Remember, the MCT covers *both* increasing sequences X_n (provided $X_n \geq Z \in L_1$) and decreasing sequences X_n (provided $X_n \leq Z \in L_1$).

Problem 4 (cont'd): Still $(\Omega, \mathcal{F}, \mathsf{P}) = \big((0,1], \mathcal{B}, \lambda\big)$

c) (4) Let $Z_n(\omega) := \omega^n$. Does Lebesgue's monotone convergence theorem apply? If so, what does it say? ◯ Yes ◯ No Reasoning:

d) (4) Let $W_n := \omega^{-1/n}$ for $n \geq 2$. Does Lebesgue's monotone convergence theorem apply? If so, what does it say? ◯ Yes ◯ No Reasoning:

e) (4) Let $S_n(\omega) := \sum_{j=1}^{n} \omega^{j-1}$ and $S(\omega) := \sum_{j=1}^{\infty} \omega^{j-1}$. Does Lebesgue's dominated convergence theorem apply? If so, what does it say? ◯ Yes ◯ No Reasoning:

Problem 5: True or false? Circle T or F. Each answer is worth 2 points. No explanations are needed, but you can give one if you think the question is ambiguous. All random variables are real on some $(\Omega, \mathcal{F}, \mathsf{P})$.

a) T F The collection $\mathcal{A} := \{A \in \mathcal{F} :\ \mathsf{P}[A] = 1\}$ is a π-system.

b) T F The collection $\mathcal{B} := \{B \in \mathcal{F} :\ \mathsf{P}[B] = 0\}$ is a λ-system.

c) T F The collection $\mathcal{C} := \{C \in \mathcal{F} :\ \mathsf{P}[C] = 0 \text{ or } 1\}$ is a σ-algebra.

d) T F $\mathsf{E}[\exp(tX)] \geq 1 + t\mathsf{E}[X]$ for any RV X and any $t \in \mathbb{R}$.

e) T F If $\mathsf{P}[X > t] = \mathsf{P}[Y > t]$ for each $t \in \mathbb{R}$ then X, Y have the same distribution.

f) T F If $\mathsf{P}[X = t] = \mathsf{P}[Y = t]$ for all $t \in \mathbb{R}$ then X, Y have the same distribution.

g) T F $X_n \to \pi$ *a.s.* if and only if $Y_n := \sin(X_n) \to 0$ *a.s.*

h) T F If X_j are independent with $\mathsf{P}[X_j = 1] = p_j = 1 - \mathsf{P}[X_j = 0]$ for some fixed $\{p_j\} \subset (0, 1)$, then $Y_n := \prod_{j=1}^n X_j \to 0$ *a.s.* as $n \to \infty$.

i) T F If $X > 0$ and $Y > 0$ are independent, then $\mathsf{E}[X/Y] = \mathsf{E}[X]/\mathsf{E}[Y]$.

j) T F If $X > 0$ and $Y > 0$ are independent then, for each $t > 0$, $\mathsf{P}[X + Y > 2t] \leq \mathsf{P}[X > t] + \mathsf{P}[Y > t]$.

Blank Worksheet

Another Blank Worksheet

Name	Notation	pdf/pmf	Range	Mean μ	Variance σ^2	
Beta	$\mathsf{Be}(\alpha,\beta)$	$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$x\in(0,1)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	
Binomial	$\mathsf{Bi}(n,p)$	$f(x)=\binom{n}{x}p^x q^{(n-x)}$	$x\in 0,\cdots,n$	$n\,p$	$n\,p\,q$	$(q=1-p)$
Exponential	$\mathsf{Ex}(\lambda)$	$f(x)=\lambda\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	$1/\lambda$	$1/\lambda^2$	
Gamma	$\mathsf{Ga}(\alpha,\lambda)$	$f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	α/λ	α/λ^2	
Geometric	$\mathsf{Ge}(p)$	$f(x)=p\,q^x$	$x\in\mathbb{Z}_+$	q/p	q/p^2	$(q=1-p)$
		$f(y)=p\,q^{y-1}$	$y\in\{1,\ldots\}$	$1/p$	q/p^2	$(y=x+1)$
HyperGeo.	$\mathsf{HG}(n,A,B)$	$f(x)=\frac{\binom{A}{x}\binom{B}{n-x}}{\binom{A+B}{n}}$	$x\in 0,\cdots,n$	$n\,P$	$n\,P\,(1-P)\frac{N-n}{N-1}$	$(P=\frac{A}{A+B})$
Logistic	$\mathsf{Lo}(\mu,\beta)$	$f(x)=\frac{e^{-(x-\mu)/\beta}}{\beta[1+e^{-(x-\mu)/\beta}]^2}$	$x\in\mathbb{R}$	μ	$\pi^2\beta^2/3$	
Log Normal	$\mathsf{LN}(\mu,\sigma^2)$	$f(x)=\frac{1}{x\sqrt{2\pi\sigma^2}}e^{-(\log x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}_+$	$e^{\mu+\sigma^2/2}$	$e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right)$	
Neg. Binom.	$\mathsf{NB}(\alpha,p)$	$f(x)=\binom{x+\alpha-1}{x}p^\alpha\,q^x$	$x\in\mathbb{Z}_+$	$\alpha q/p$	$\alpha q/p^2$	$(q=1-p)$
		$f(y)=\binom{y-1}{y-\alpha}p^\alpha\,q^{y-\alpha}$	$y\in\{\alpha,\ldots\}$	α/p	$\alpha q/p^2$	$(y=x+\alpha)$
Normal	$\mathsf{No}(\mu,\sigma^2)$	$f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}$	μ	σ^2	
Pareto	$\mathsf{Pa}(\alpha,\epsilon)$	$f(x)=(\alpha/\epsilon)(1+x/\epsilon)^{-\alpha-1}$	$x\in\mathbb{R}_+$	$\frac{\epsilon}{\alpha-1}$ if $\alpha>1$	$\frac{\epsilon^2\alpha}{(\alpha-1)^2(\alpha-2)}$ if $\alpha>2$	
		$f(y)=\alpha\,\epsilon^\alpha/y^{\alpha+1}$	$y\in(\epsilon,\infty)$	$\frac{\epsilon\,\alpha}{\alpha-1}$ if $\alpha>1$	$\frac{\epsilon^2\alpha}{(\alpha-1)^2(\alpha-2)}$ if $\alpha>2$	$(y=x+\epsilon)$
Poisson	$\mathsf{Po}(\lambda)$	$f(x)=\frac{\lambda^x}{x!}e^{-\lambda}$	$x\in\mathbb{Z}_+$	λ	λ	
Snedecor F	$F(\nu_1,\nu_2)$	$f(x)=\frac{\Gamma(\frac{\nu_1+\nu_2}{2})(\nu_1/\nu_2)^{\nu_1/2}}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})}\times$	$x\in\mathbb{R}_+$	$\frac{\nu_2}{\nu_2-2}$ if $\nu_2>2$	$\left(\frac{\nu_2}{\nu_2-2}\right)^2\frac{2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)}$ if $\nu_2>4$	
		$x^{\frac{\nu_1-2}{2}}\left[1+\frac{\nu_1}{\nu_2}x\right]^{-\frac{\nu_1+\nu_2}{2}}$				
Student t	$t(\nu)$	$f(x)=\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}}[1+x^2/\nu]^{-(\nu+1)/2}$	$x\in\mathbb{R}$	0 if $\nu>1$	$\frac{\nu}{\nu-2}$ if $\nu>2$	
Uniform	$\mathsf{Un}(a,b)$	$f(x)=\frac{1}{b-a}$	$x\in(a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	
Weibull	$\mathsf{We}(\alpha,\beta)$	$f(x)=\alpha\beta\,x^{\alpha-1}\,e^{-\beta\,x^\alpha}$	$x\in\mathbb{R}_+$	$\frac{\Gamma(1+\alpha^{-1})}{\beta^{1/\alpha}}$	$\frac{\Gamma(1+2/\alpha)-\Gamma^2(1+1/\alpha)}{\beta^{2/\alpha}}$	