Midterm Examination II

STA 711: Probability & Measure Theory

Thursday, 2019 Nov 14, 1:25 – 2:40pm

This is a closed-book exam. You may use a single sheet of prepared notes, if you wish, but you may not share materials.

If a question seems ambiguous or confusing, *please* ask me to clarify it. Unless a problem states otherwise, you must **show** your **work**. There is a blank worksheet at the end of the test if you need more room for this, and also a pdf/pmf sheet.

It is to your advantage to write your solutions as clearly as possible, and to box answers I might not find.

For full credit, answers must be given in **closed form** with no unevaluated sums, integrals, maxima, unreduced fractions. Wherever possible, **Simplify**.

Good luck!

Print Name Clearly: ______________________________

1.	/20
2.	/20
3.	/20
4.	/20
5.	/20
Total:	/100

Version a

Problem 1: Let X and Y be real-valued random variables with characteristic functions (ch.f.s) $\phi(\theta) := \mathsf{E}[\exp(i\theta X)]$ and $\psi(\theta) := \mathsf{E}[\exp(i\theta Y)]$ for $\theta \in \mathbb{R}$.

a) (5) If the random variable X takes the values -1, 0, and 1 with equal probabilities $\mathsf{P}[X = x] = 1/3$, $x \in \{-1, 0, 1\}$, find $\phi(\theta)$. Simplify! $\phi(\theta) =$

b) (5) If a ch.f. is $\psi(\theta) = c[1 + \cos(2\theta)]$, what is the value of $c \in \mathbb{R}$? Simplify! $c =$

Problem 1 (cont'd): Still Y is an RV with ch.f. $\psi(\theta) := \mathsf{E}[\exp(i\theta Y)]$.

c) (5) If the ch.f. of Y is $\psi(\theta) = c[1+\cos(2\theta)]$, what are the possible values of Y and their probabilities? Use as many columns as you need (perhaps fewer than six):

$y =$						
$\mathsf{P}[Y = y] =$						

d) (5) Prove or disprove: A ch.f. $\chi(\theta) = \mathsf{E}[\exp(i\theta Z)]$ takes on only real values if and only if Z has a symmetric distribution, *i.e.*, if Z and $-Z$ have the same distribution.

Problem 2: Let $\Omega = \{1, 2, 3, 4, 5\}$ and define a probability measure P on the σ-algebra $\mathcal{F} := 2^\Omega$ by extending $\mathsf{P}\big(\{\omega\}\big) = \omega/15$. Define a random variable X, an event A, and a σ-algebra $\mathcal{G}$ by

$$X(\omega) := \omega \qquad A := \{1, 3\} \qquad \mathcal{G} := \sigma\big(\{1, 2\}, \{3\}, \{4, 5\}\big).$$

a) (6) Find: $\mathsf{P}[A \mid \mathcal{G}] =$

b) (6) Find: $\mathsf{E}[X \mid \mathcal{G}] =$

c) (4) Find: $\mathsf{E}[X \mid \mathcal{F}] =$

d) (4) Define a new probability measure by extending $Q\big(\{\omega\}\big) = q(\omega)$ for the function $q : \Omega \to \mathbb{R}_+$ given below. Complete the specification of Q to ensure that $\sigma(A)$ and $\mathcal{G}$ are independent on the probability space $(\Omega, \mathcal{F}, Q)$:

$\omega =$	1	2	3	4	5
$q(\omega) =$			0.2	0	0

Problem 3: Let $\{U_n\} \overset{\text{iid}}{\sim} \mathsf{Un}(0,1)$ be independent uniformly-distrbuted RVs and set $X_n := 1/\sqrt{nU_n}$ for $n \in \mathbb{N}$.

a) (5) Does $X_n \to 0$ *pr.*? ◯ Yes ◯ No Proof:

b) (5) Does $X_n \to 0$ *a.s.*? ◯ Yes ◯ No Proof:

Problem 3 (cont'd): Still $\{U_n\} \overset{\text{iid}}{\sim} \mathsf{Un}(0,1)$ and $X_n := 1/\sqrt{nU_n}$.

c) (5) Does $X_n \to 0$ in L_1? ◯ Yes ◯ No Proof:

d) (5) Does $X_n \to 0$ in L_2? ◯ Yes ◯ No Proof:

Problem 4: Let P and Q be two probability measures on the same space $(\Omega, \mathcal{F})$, and let $\{X_n\}$ be $\mathcal{F}\backslash\mathcal{B}$-measurable functions from Ω to $\mathbb{R}$. The random variables $\{X_i\}$ take only the values 0 and 1, and are independent under both P and Q, but with slightly different distributions:

$$\begin{array}{ll} \mathsf{P}[X_n = 1] = 0.50 & \mathsf{Q}[X_n = 1] = 0.60 \\ \mathsf{P}[X_n = 0] = 0.50 & \mathsf{Q}[X_n = 0] = 0.40. \end{array}$$

Set $S_n := \sum_{j \le n} X_j$, the partial sums, and $\bar{X}_n := S_n/n$, the sample average. Fix $\epsilon > 0$.

a) (5) What bound does Chebychev's inequality give for:
$\mathsf{P}[\ |\bar{X}_n - 0.50| > \epsilon\ ] \le$

b) (5) What bound does Hoeffding's inequality[1] give for:
$\mathsf{P}[\ |\bar{X}_n - 0.50| > \epsilon\ ] \le$

c) (5) Use either a) or b) (whichever one helps) and the Borel/Cantelli lemma to prove that $\mathsf{P}[\bar{X}_n \to 0.50] = 1$.

[1]Hoeffding: If $\{X_j\} \subset L_\infty$ are indep. with $a_j \le X_j \le b_j$, $S_n := \sum_{j \le n} X_j$, $\mu_n := \mathsf{E}[S_n]$, then $(\forall c > 0)$ $\mathsf{P}[S_n - \mu_n > c] \le \exp\big(-2c^2 / \sum_{j \le n} |b_j - a_j|^2\big)$.

Problem 4 (cont'd): Still $\{X_n\}$ are iid Bernoulli RVs on $(\Omega, \mathcal{F})$ under both P and Q, with success probabilities 0.50 and 0.60 respectively.

d) (5) Show that P and Q are mutually singular ($\mathsf{P} \perp \mathsf{Q}$) on $(\Omega, \mathcal{F})$ by finding disjoint events $A, B \in \mathcal{F}$ with $\mathsf{P}[A] = 1$ and $\mathsf{Q}[B] = 1$. What theorem supports your choice?

e) (XC) Let $\mathcal{F}_n := \sigma\{X_1, \cdots, X_n\}$. Show that $\mathsf{P} \equiv \mathsf{Q}$ on $(\Omega, \mathcal{F}_n)$ by finding the Radon-Nikodym derivative $H := d\mathsf{Q}/d\mathsf{P}$ explicitly in terms of S_n. What happens as $n \to \infty$?
$H(\omega) =$

Problem 5: True or false? Circle one, for 2 points each. No explanations are needed. All random variables are real on the same space $(\Omega, \mathcal{F}, \mathsf{P})$.

a) T F If X is $\mathbb{R}$-valued then $\cup_{n\in\mathbb{N}}\{|X| \le n\}$ has probability one.

b) T F If $\{X_n\} \subset L_1(\Omega, \mathcal{F}, \mathsf{P})$ and $\{X_n\}$ are independent then $\sum_n \mathsf{E}[X_n] = \mathsf{E}\big[\sum_n X_n\big]$.

c) T F If $\{X, Y\}$ are iid & *a.s.* positive then $\mathsf{E}[X/(X+Y)] = 1/2$.

d) T F If $A \subset \Omega$ is countable then $\mathsf{P}[A] = 0$.

e) T F If $\{X_n\}$ converge *a.s.* to a limit X and satisfy $\mathsf{P}\big[|X_n| \le n\big] = 1$, then $X_n \to X$ in L_1.

f) T F If $\mathsf{E}|X_n|^2 \to 0$ then also $\mathsf{E}\sqrt{|X_n|} \to 0$.

g) T F If $X \perp\!\!\!\perp Y$ and $X > 0$ and $Y > 0$ then $\mathsf{E}[X/Y] = \mathsf{E}[X]\,\mathsf{E}[1/Y]$.

h) T F If $\{X_n\} \subset L_1$ are iid then they are UI.

i) T F If X and Y are in L_p then $|X| \vee |Y| \equiv \max\big(|X|, |Y|\big) \in L_p$.

j) T F If $\{X_n\}$ are iid with $\mathsf{P}[X_n > x] = 1/(1+x)$ for $n \in \mathbb{N}$ and $x > 0$, then $X_{*n} := \min\{X_1, \cdots, X_n\} \in L_p$ for $0 < p < n$.

Blank Worksheet

Name	Notation	pdf/pmf	Range	Mean μ	Variance σ^2	
Beta	$\mathsf{Be}(\alpha,\beta)$	$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$	$x \in (0,1)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	
Binomial	$\mathsf{Bi}(n,p)$	$f(x) = \binom{n}{x} p^x q^{(n-x)}$	$x \in 0, \cdots, n$	$n\,p$	$n\,p\,q$	$(q = 1-p)$
Exponential	$\mathsf{Ex}(\lambda)$	$f(x) = \lambda\, e^{-\lambda x}$	$x \in \mathbb{R}_+$	$1/\lambda$	$1/\lambda^2$	
Gamma	$\mathsf{Ga}(\alpha,\lambda)$	$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1}\, e^{-\lambda x}$	$x \in \mathbb{R}_+$	α/λ	α/λ^2	
Geometric	$\mathsf{Ge}(p)$	$f(x) = p\, q^x$	$x \in \mathbb{Z}_+$	q/p	q/p^2	$(q = 1-p)$
		$f(y) = p\, q^{y-1}$	$y \in \{1, ...\}$	$1/p$	q/p^2	$(y = x+1)$
HyperGeo.	$\mathsf{HG}(n,A,B)$	$f(x) = \frac{\binom{A}{x}\binom{B}{n-x}}{\binom{A+B}{n}}$	$x \in 0, \cdots, n$	$n\,P$	$n\,P\,(1-P)\frac{N-n}{N-1}$	$(P = \frac{A}{A+B})$
Logistic	$\mathsf{Lo}(\mu,\beta)$	$f(x) = \frac{e^{-(x-\mu)/\beta}}{\beta[1+e^{-(x-\mu)/\beta}]^2}$	$x \in \mathbb{R}$	μ	$\pi^2\beta^2/3$	
Log Normal	$\mathsf{LN}(\mu,\sigma^2)$	$f(x) = \frac{1}{x\sqrt{2\pi\sigma^2}} e^{-(\log x-\mu)^2/2\sigma^2}$	$x \in \mathbb{R}_+$	$e^{\mu+\sigma^2/2}$	$e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right)$	
Neg. Binom.	$\mathsf{NB}(\alpha,p)$	$f(x) = \binom{x+\alpha-1}{x} p^\alpha\, q^x$	$x \in \mathbb{Z}_+$	$\alpha q/p$	$\alpha q/p^2$	$(q = 1-p)$
		$f(y) = \binom{y-1}{y-\alpha} p^\alpha\, q^{y-\alpha}$	$y \in \{\alpha, ...\}$	α/p	$\alpha q/p^2$	$(y = x+\alpha)$
Normal	$\mathsf{No}(\mu,\sigma^2)$	$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$	$x \in \mathbb{R}$	μ	σ^2	
Pareto	$\mathsf{Pa}(\alpha,\epsilon)$	$f(x) = (\alpha/\epsilon)(1+x/\epsilon)^{-\alpha-1}$	$x \in \mathbb{R}_+$	$\frac{\epsilon}{\alpha-1}$ if $\alpha > 1$	$\frac{\epsilon^2\alpha}{(\alpha-1)^2(\alpha-2)}$ if $\alpha > 2$	
		$f(y) = \alpha\,\epsilon^\alpha/y^{\alpha+1}$	$y \in (\epsilon,\infty)$	$\frac{\epsilon\,\alpha}{\alpha-1}$ if $\alpha > 1$	$\frac{\epsilon^2\alpha}{(\alpha-1)^2(\alpha-2)}$ if $\alpha > 2$	$(y = x+\epsilon)$
Poisson	$\mathsf{Po}(\lambda)$	$f(x) = \frac{\lambda^x}{x!} e^{-\lambda}$	$x \in \mathbb{Z}_+$	λ	λ	
Snedecor F	$F(\nu_1,\nu_2)$	$f(x) = \frac{\Gamma(\frac{\nu_1+\nu_2}{2})(\nu_1/\nu_2)^{\nu_1/2}}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} \times x^{\frac{\nu_1-2}{2}} \left[1+\frac{\nu_1}{\nu_2}x\right]^{-\frac{\nu_1+\nu_2}{2}}$	$x \in \mathbb{R}_+$	$\frac{\nu_2}{\nu_2-2}$ if $\nu_2 > 2$	$\left(\frac{\nu_2}{\nu_2-2}\right)^2 \frac{2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)}$ if $\nu_2 > 4$	
Student t	$t(\nu)$	$f(x) = \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}}[1+x^2/\nu]^{-(\nu+1)/2}$	$x \in \mathbb{R}$	0 if $\nu > 1$	$\frac{\nu}{\nu-2}$ if $\nu > 2$	
Uniform	$\mathsf{Un}(a,b)$	$f(x) = \frac{1}{b-a}$	$x \in (a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	
Weibull	$\mathsf{We}(\alpha,\beta)$	$f(x) = \alpha\beta\, x^{\alpha-1}\, e^{-\beta\, x^\alpha}$	$x \in \mathbb{R}_+$	$\frac{\Gamma(1+\alpha^{-1})}{\beta^{1/\alpha}}$	$\frac{\Gamma(1+2/\alpha)-\Gamma^2(1+1/\alpha)}{\beta^{2/\alpha}}$	