Sta 711: Homework 2

σ-Algebras

1. A *partition* of a measurable space $(\Omega, \mathcal{F})$ is a (finite or countable) collection of disjoint and non-empty events $\Lambda_j \in \mathcal{F}$ with $\Omega = \cup_j \Lambda_j$. Let $\{A, B\} \subset \mathcal{F}$ be two non-empty disjoint events with $A \cup B \neq \Omega$. Enumerate the elements of the smallest partition $\mathcal{P} := \mathcal{P}(A, B)$ containing A, B and of the σ-algebra $\sigma(\mathcal{P})$ it generates. How many elements are in $\sigma(\mathcal{P})$?

2. Consider two classes of sets $\mathcal{G} \subset \mathcal{F}$ and $\mathcal{H} \subset \mathcal{F}$ on a measurable space $(\Omega, \mathcal{F})$. Suppose that for every event $A \in \mathcal{H}$ there exists events $\{B_i\} \subset \mathcal{G}$ such that $A = \cup_i B_i$. Further suppose that for every $B \in \mathcal{G}$ there exists a set $A \in \mathcal{H}$ such that $A^c = B$. Show that the sigma-algebras $\sigma(\mathcal{G})$ and $\sigma(\mathcal{H})$ they generate are equal.

Null sets.

3. Let $\{A_n, n \in \mathbb{N}\}$ be events with $\mathsf{P}(A_n) = 1$. Prove that $\mathsf{P}(\cap_{n=1}^{\infty} A_n) = 1$.

4. Now consider uncountably many events $\{B_\alpha\}$, all with $\mathsf{P}(B_\alpha) = 1$. Does it follow necessarily that $\mathsf{P}(\cap_\alpha B_\alpha) = 1$? Give a proof or a counter example.

5. Let $n \in \mathbb{N}$ and let $\{C_k\}$ be a collection of n events such that $\sum_{k=1}^{n} \mathsf{P}(C_k) > n - 1$. Show that $\mathsf{P}(\cap_{k=1}^{n} C_k) > 0$.

Distribution functions and continuity.

6. Give an example[1] of a real-valued function on $\mathbb{R}$ which is continuous, but **not** uniformly continuous.

7. Let G be a continuous distribution function on $\mathbb{R}$. Show[2] that G is in fact *uniformly* continuous, *i.e.*, that $(\forall \epsilon > 0)(\exists \delta > 0)(\forall x, y \in \mathbb{R})\ |x - y| < \delta \Rightarrow |G(x) - G(y)| < \epsilon$.

8. Show that any distribution function F on $\mathbb{R}$ can have *at most countably many* discontinuities. Hint: Consider the open intervals $(F(x-), F(x))$ for discontinuity points x.

9. Let $\{A_n\}_{n \in \mathbb{N}} \subset \mathcal{F}$ be *increasing* in the sense that each $A_n \subset A_{n+1}$. Prove that $\mathsf{P}(A_n) \to \mathsf{P}\big(\cup_{n \in \mathbb{N}} A_n\big)$, a property called "continuity". What happens for $\{B_n\} \subset \mathcal{F}$ with $B_n \supset B_{n+1}$?

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[1] and, of course, *prove* that your example satisfies the criteria.

[2] Hint: Consider points $\{x_i\}$ for which $G(x_i) = i/n$ for $1 \le i < n$. Must these exist? If they do, are they determined uniquely? Does that matter? Draw a graph!

π- & λ- systems.

10. Consider the following collecton of subsets of the real line:

$$\mathcal{B} = \{(-\infty, b] : \ b \in \mathbb{R}\}$$

   (a) Show that $\mathcal{B}$ is a π- system, but not a λ system.

   (b) What is the λ- system generated by $\mathcal{B}$? Why?

11. Consider the following collections of subsets of the unit square $\Omega = (0,1]^2 \subset \mathbb{R}^2$:

$$\mathcal{A} = \{(a,b] \times (c,d] : \ 0 \le a \le b \le 1, \ 0 \le c \le d \le 1\}$$

   (a) Is $\mathcal{A}$ a π- system? Why or why not?

   (b) Is $\mathcal{A}$ a λ- system? Why or why not?

π - systems and fields.

Let $\mathcal{C}$ be a nonempty collection of subsets of a space Ω.

12. Let $\mathcal{F}(\mathcal{C})$ be the smallest field containing $\mathcal{C}$. Show that for each $B \in \mathcal{F}(\mathcal{C})$ there exists a *finite* subcollection $\mathcal{C}_B \subseteq \mathcal{C}$ for which $B \in \mathcal{F}(\mathcal{C}_B)$. Note $\mathcal{C}_B$ may depend on B. You should be able to do this without using (13). Suggestion: Let $\mathcal{G} := \{B \in \mathcal{F}(\mathcal{C}) : \ B \in \mathcal{F}(C_B)$ for some finite $\mathcal{C}_B \subset \mathcal{C}\}$. How can you show $\mathcal{F}(\mathcal{C}) \subseteq \mathcal{G}$?

13. The smallest field $\mathcal{F}(\mathcal{C})$ containing any nonempty collection $\mathcal{C} \subset 2^\Omega$ is precisely:

$$\mathcal{G} := \left\{B : \ B = \cup_{i=1}^m B_i, \quad B_i = \cap_{j=1}^{n_i} A_{ij} \text{ for some } m \in \mathbb{N}, \ \{n_i\} \subset \mathbb{N}\right\}$$

with each $A_{ij} \in \mathcal{C}$ or $A_{ij}^c \in \mathcal{C}$, and with the m sets $\{B_i\}$ disjoint. Thus every set in the *field* $\mathcal{F}(\mathcal{C})$ can be represented explicitly (interestingly, this is impossible for σ-fields). To prove this (which you do *not* have to do), one would have to show five things:

(a) $\Omega \in \mathcal{G}$	(d) $\mathcal{C} \subset \mathcal{G}$
(b) $A \in \mathcal{G} \Rightarrow A^c \in \mathcal{G}$	(e) $\mathcal{C} \subset \mathcal{H}$ and $\mathcal{H}$ a field $\Rightarrow \mathcal{G} \subset \mathcal{H}$
(c) $A, B \in \mathcal{G} \Rightarrow A \cup B \in \mathcal{G}$	

Items (a,b,c) show $\mathcal{G}$ is a field; (d) shows it contains $\mathcal{C}$; and (e) shows it's smallest. Verify just (a), (d), and (e). Conditions (b) and (c) are routine but tedious.

14. Show that if two probability measures P_1, P_2 agree on a π system $\mathcal{C}$, then they must also agree on the field $\mathcal{F}(\mathcal{C})$ generated by $\mathcal{C}$. Don't just quote the result from the text.

15. Find two probability measures P_1, P_2 on some measurable space $(\Omega, \mathcal{F})$ that agree on a nonempty collection of subsets $\mathcal{C}$, but *not* on $\mathcal{F}(\mathcal{C})$. Obviously from (14) above $\mathcal{C}$ cannot be a π-system. Hint: It's enough to have an outcome space Ω with just four points, and $\mathcal{C} = \{A, B\} \subset 2^\Omega$ with just two events; give Ω, A, B, P_1, and P_2 explicitly. Would Ω with three points be enough?

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