Sta 711: Homework 4

Expectation

Feel free to use the result of one problem in your solution to a subsequent problem.

1. Let $X := (X_1, X_2)$ be distributed uniformly over the triangle in $\mathbb{R}^2$ with vertices $\{(-1,0),(1,0),(0,1)\}$. Evaluate $\mathsf{E}(X_1 + X_2)$ (no need to approximate by simple functions, just find the value and show how you did it).

2. Let $X \geq 0$ be a nonnegative random variable on $(\Omega, \mathcal{F}, \mathsf{P})$ and, for $n \in \mathbb{N}$, set

$$X_n(\omega) := \min\big(2^n, 2^{-n}\lfloor 2^n X(\omega)\rfloor\big).$$

For fixed X and n, prove that X_n is simple (how many distinct values can it take on?) and $X_n \nearrow X$ as $n \to \infty$. Note you must show *both* monotonicity and convergence. For $\omega \in \Omega$ and $\epsilon > 0$, how big must n be to ensure $|X - X_n| < \epsilon$?

3. Suppose $X \in L_1(\Omega, \mathcal{F}, \mathsf{P})$, *i.e.*, $\mathsf{E}|X| < \infty$. Show that[1]

$$\int_{|X|>m} X\,d\mathsf{P} \to 0 \qquad \text{as} \quad m \to \infty.$$

4. Let $\{A_n\}$ denote a sequence of events such that $\mathsf{P}(A_n) \to 0$ as $n \to \infty$ and let $X \in L_1$. Show that

$$\mathsf{E}[X\mathbf{1}_{A_n}] = \int_{A_n} X\,d\mathsf{P} \to 0$$

Hint: Use problem 3. Warning: these conditions do *not* imply that $X(\omega)\mathbf{1}_{A_n}(\omega) \to 0$.

5. Fix a probability space $(\Omega, \mathcal{F}, \mathsf{P})$ and define a distance measure d on $\mathcal{F}$ by $\mathrm{d}(A,B) := \mathsf{P}(A\Delta B)$ where (as usual) $A\Delta B := (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$ denotes the symmetric difference. Show that, if $\{A_n\} \subset \mathcal{F}$ and $A \in \mathcal{F}$ satisfy $\mathrm{d}(A_n, A) \to 0$, then

$$\int_{A_n} X\,d\mathsf{P} \to \int_A X\,d\mathsf{P}$$

for every $X \in L_1(\Omega, \mathcal{F}, \mathsf{P})$. Hint: Use problem 4.

[1] The "expectation of a random variable X over an event A" can be written in many ways, including $\int_A X\,d\mathsf{P} = \mathsf{E}[X\mathbf{1}_A] = \int_A X(\omega)\mathsf{P}(d\omega)$.

Convergence Theorems

6. Let $X \geq 0$ be a non-negative random variable. Define sequences of random variables X_n and of extended real numbers $0 \leq S_n \leq \infty$ for positive integers $n \in \mathbb{N}$ by:

$$X_n := \sum_{k=0}^{\infty} \frac{k}{2^n} \mathbf{1}_{\{k<2^n X \leq k+1\}} \qquad S_n := \mathsf{E} X_n = \sum_{k=0}^{\infty} \frac{k}{2^n} \mathsf{P} \left\{ \frac{k}{2^n} < X \leq \frac{k+1}{2^n} \right\}$$

Is X_n "simple"? What is $\lim_{n\to\infty} S_n$? Justify your answers.

7. Define a sequence of random variables on $(\Omega, \mathcal{F}, \mathsf{P}) = ((0,1], \mathcal{B}, \lambda)$ by

$$X_n := \frac{n}{\log(n+1)} \mathbf{1}_{(0,\frac{1}{n}]} \qquad n \in \mathbb{N}.$$

Show that $\mathsf{P}[X_n \to 0] = 1$, and that $\mathsf{E}(X_n) \to 0$. Also show that the Dominated Convergence Theorem does not apply to this example. Why?

8. Let $\{Y_n\}$ be a sequence of random variables for $n \in \mathbb{N}$ with

$$\mathsf{P}(Y_n = -n^3) = \mathsf{P}(Y_n = +n^3) = \frac{1}{2n^2}, \qquad \mathsf{P}(Y_n = 0) = 1 - \frac{1}{n^2}$$

One can (but you don't have to) use the Borel-Cantelli lemma to show that $Y_n \to 0$ *a.s.* Compute $\lim_{n\to\infty} \mathsf{E}(Y_n)$ and $\lim_{n\to\infty} \mathsf{E}(|Y_n|)$. Is the Dominated Convergence Theorem applicable? Why or why not?

9. Let $\{X_n\}, X$ be random variables with $0 \leq X_n \to X$. If $\sup_n \mathsf{E}(X_n) \leq K < \infty$, show that $X \in L_1$ and $\mathsf{E}(X) \leq K$. Does $X_n \to X$ in L_1?

Domination

10. Let $\{X_n\}$ be a sequence of random variables. We have seen that $\{X_n\}$ is dominated by some integrable Y, *i.e.*, $|X_n| \leq Y$ and $\mathsf{E}Y < \infty$, if and only if

$$\mathsf{E}\left(\sup_{n\in\mathbb{N}} |X_n|\right) < \infty \tag{1}$$

Thus, (1) is exactly equivalent to domination in Lebesgue's sense (but Lebesgue's domination is often easier to verify). Does the condition

$$\sup_{n\in\mathbb{N}} \mathsf{E}\big(|X_n|\big) < \infty \tag{2}$$

imply (1)? Or is it implied *by* (1)? For each direction (1⇒2 and 2⇒1), give either a proof or a counter-example.

Last edited: September 15, 2020