Sta 711: Homework 6

Independence

1. Let $\{B_i\}$ be independent events. For $n \in \mathbb{N}$ show that

$$\mathsf{P}\left(\bigcup_{i=1}^{n} B_i\right) = 1 - \prod_{i=1}^{n}\left[1 - \mathsf{P}(B_i)\right] \geq 1 - \exp\left\{-\sum_{i=1}^{n}\mathsf{P}(B_i)\right\}$$

and conclude that $\mathsf{P}\left[\cup_{i=1}^{\infty} B_i\right] = 1$ if each $\mathsf{P}[B_i] \geq \epsilon$ for some $\epsilon > 0$. Show that this conclusion would be false without the assumption of independence.

2. If $\{A_n, n \in \mathbb{N}\}$ is a sequence of events such that $\mathsf{P}[A_n] = 1/3$ for each n and

$$(\forall n \neq m \in \mathbb{N}) \quad \mathsf{P}(A_n \cap A_m) = \mathsf{P}(A_n)\mathsf{P}(A_m),$$

does it follow that the events $\{A_n\}$ are independent? Give a proof or counter-example. Note $1/3 \neq 1/2$.

3. Show that a random variable Y is independent of itself if and only if, for some constant $c \in \mathbb{R}$, $\mathsf{P}[Y = c] = 1$.
Let $f : \mathbb{R} \to \mathbb{R}$ be Borel measurable, and X a non-constant random variable. Can $Y := f(X)$ and X be independent? Explain your answer.

4. Give an example to show that an event $A \in \mathcal{F}$ may be independent of each B in some collection $\mathcal{C} \subset \mathcal{F}$ of events, but *not* independent of $\sigma(\mathcal{C})$. Prove this is impossible if $\mathcal{C}$ is a π-system (*i.e.*, in that case A must be independent of $\sigma(\mathcal{C})$).

5. Give a simple example to show that two random variables on the same space $(\Omega, \mathcal{F})$ may be independent according to one probability measure P_1 but dependent with respect to another P_2.

Fubini's Theorem

6. Let $X \geq 0$ be a positive random variable and $\alpha > 0$. Show that

$$\mathsf{E}(X^\alpha) = \alpha \int_0^\infty t^{\alpha-1}\mathsf{P}(X > t)dt.$$

Note that the distribution $\mu(dx)$ of X need not be absolutely continuous (so X may not have a pdf). Where did you use Fubini's theorem? Why is it applicable?

7. Define measure spaces $(\Omega_i, \mathcal{F}_i, \mu_i)$, for $i = 1, 2$ as follows. Let each $\Omega_i := (0, 1]$, the unit interval, with σ-algebras

$$\mathcal{F}_1 = \mathcal{B} = \text{ Borel sets of (0,1]} \qquad \mathcal{F}_2 = 2^\Omega = \text{ All subsets of (0,1]},$$

and let $\mu_1 = \lambda$ be Lebesgue measure and μ_2 counting measure— so $\mu_1(A)$ is the length of any Borel set $A \in \mathcal{F}_1$ and $\mu_2(B)$ is the cardinality of $B \subset (0,1]$. Define:

$$f(x,y) := \mathbf{1}_{x=y}(x,y),$$

the indicator of the diagonal in $\Omega_1 \times \Omega_2 = (0,1]^2$. Set

$$I_1 := \int_{\Omega_1} \left[\int_{\Omega_2} f(x,y)\mu_2(dy) \right] \mu_1(dx) \qquad I_2 := \int_{\Omega_2} \left[\int_{\Omega_1} f(x,y)\mu_1(dx) \right] \mu_2(dy)$$

Compute I_1 and I_2. Is $I_1 = I_2$? Are the measures μ_1 and μ_2 σ-finite? Why doesn't Fubini's theorem hold here?

8. This problem is a probabilistic version of the familiar integration-by-parts formula from calculus. Suppose F and G are two distribution functions with no common points of discontinuity on an interval $(a,b]$. Show that

$$\int_{(a,b]} G(x)F(dx) = F(b)G(b) - F(a)G(a) - \int_{(a,b]} F(x)G(dx)$$

where "$G(dx)$" denotes the measure on $(\mathbb{R}, \mathcal{B})$ with DF $G(x)$. Show that the formula fails if F and G have common discontinuities. Hint: Let X and Y be independent RVs with DFs F and G.

Zero-One Laws

9. Let $\{X_n\}$ be a sequence of Bernoulli random variables with

$$\mathsf{P}(X_n = 1) = n^{-p} \qquad \mathsf{P}(X_n = 0) = 1 - n^{-p}$$

for some $p > 0$. For $p = 2$ show that the partial sum

$$S_n := \sum_{k=1}^{n} X_k$$

converges almost-surely, whether or not the $\{X_n\}$ are independent. If the $\{X_n\}$ are independent, for which $p > 0$, does S_n converge? Why?

10. Let $\{X_n\}$ be an iid sequence of random variables with a non-degenerate distribution (*i.e.*, for some $B \in \mathcal{B}$, $0 < \mathsf{P}[X_n \in B] < 1$). Show that

$$\mathsf{P}[\omega : \ X_n(\omega) \text{ converges}] = 0$$

11. Use the Borel-Cantelli lemma to prove that for any sequence of real-valued random variables $\{X_n\}$ (not necessarily independent or identically-distributed), there exist constants $c_n \to \infty$ such that

$$\mathsf{P}\left(\lim_{n\to\infty} \frac{X_n}{c_n} = 0 \right) = 1.$$

Give a careful description of how you choose c_n (it will depend on the distributions of the X_n). Find a suitable sequence $\{c_n\}$ explicitly for an iid sequence $\{X_n\} \overset{\text{iid}}{\sim} \mathsf{Ex}(1)$ of unit-rate exponentially-distributed random variables to ensure that $X_n/c_n \to 0$ almost surely.

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