Sta 711: Homework 7

Almost-sure Convergence

1. Let $\{X_n\}$ be a monotonically increasing sequence of RVs such that $X_n \to X$ in probability (*pr.*). Show that $X_n \to X$ almost surely (*a.s.*)

2. Let $\{X_n\}$ be any sequence of RVs. Show that $X_n \to X$ *a.s.* if and only if

$$\sup_{k\ge n} |X_k - X| \to 0 \qquad pr.$$

3. Let $\{X_n\}$ be an arbitrary sequence of RVs and set $S_n := \sum_{i=1}^n X_i$. Show that $X_n \to 0$ *a.s.* implies that $S_n/n \to 0$ *a.s.*

In-probability Convergence

4. Let $\{X_n\} \subset L_2$ be independent and identically distributed. For each $\delta > 0$ show that $n\mathsf{P}\left[|X_1| > \delta\sqrt{n}\right] \to 0$. Use this to show that the maximum $\bigvee_{i=1}^n |X_i|/\sqrt{n} \to 0$ *pr.* Thus, the maximum of n iid L_2 random variables grows slower than $\sqrt{n}$.

5. For random variables X, Y define

$$\rho(X,Y) := \inf\{\epsilon > 0 : \mathsf{P}[|X-Y| \ge \epsilon] \le \epsilon\}.$$

   The function ρ is a metric (you do *not* have to prove that), *i.e.*, it's non-negative, symmetric, satisfies the triangle inequality, and vanishes if and only if $X = Y$ *a.s.* Show that $X_n \to X$ *pr.* if and only if $\rho(X_n, X) \to 0$. Thus, convergence in probability is metrizable.[1]

6. Let P and Q be two probability measures on $(\Omega, \mathcal{F})$ that are "equivalent" in the sense that $\mathsf{P}[A] = 0 \;\Leftrightarrow\; \mathsf{Q}[A] = 0$ for each $A \in \mathcal{F}$. Show that if a sequence X_n of random variables converges to zero in probability under P then it does so under Q too.

L_p Convergence

7. Find a sequence of RVs $\{X_n\} \subset L_2$ which converge in L_1 but not in L_2.

8. Let $(\Omega, \mathcal{F}, \mathsf{P}) := \big((0,1], \mathcal{B}, \lambda\big)$ be the unit interval with Borel sets and Lebesgue measure and define $X_n(\omega) := \omega^n$ for $n \in \mathbb{N}$, $\omega \in \Omega$. For what $p \in [1, \infty]$, does the sequence $\{X_n\}$ converge in L_p ? To what limit? Explain your answer.

9. Verify Hölder's inequality for $p = 1$, $q = \infty$ and all random variables X, Y:

$$\mathsf{E}|XY| \le \|X\|_1 \|Y\|_\infty$$

   where $\|Y\|_\infty := \sup\{c < \infty : \;\mathsf{P}\big[|Y| > c\big] > 0\}$.

10. Verify Minkowski's inequality for $p = \infty$ and all random variables X, Y:

$$\|X+Y\|_\infty \le \|X\|_\infty + \|Y\|_\infty$$

[1] Many other metrics would work too— like $\mathsf{E}\big[|X-Y| \wedge 1\big]$ or $\mathsf{E}\big[|X-Y|/(1+|X-Y|)\big]$.

Uniform Integrability (UI)

11. Fix $p > 0$ and set $X_n := n^p \mathbf{1}_{\{0<\omega\le 1/n\}}$ on $(\Omega, \mathcal{F}, \mathsf{P})$ with $\Omega = (0,1]$, $\mathcal{F} = \mathcal{B}(\Omega)$, and $\mathsf{P} = \lambda$. Show explicitly that $\{X_n\}$ is UI for $p < 1$ and not for $p \ge 1$, by verifying that $\mathsf{E}\big[X_n \mathbf{1}_{\{X_n > t\}}\big]$ converges to zero uniformly as $t \to \infty$ for $p < 1$ and not for $p \ge 1$.

12. Let $\{X_n\}$ be an iid sequence of L_1 random variables and set $S_n := \sum_{i=1}^n X_i$. Show that the sequence of random variables $\{\bar{X}_n\}$ defined by $\bar{X}_n := S_n/n$ is UI.

13. Let $\{X_n\}$ be iid and L_1. Show[2]:

$$\mathsf{P}\left(\lim_{n\to\infty} \frac{X_n}{n} = 0\right) = 1.$$

14. If $\{X_n\}$ and $\{Y_n\}$ are UI, show that so is $\{X_n + Y_n\}$.

15. Let $\phi(x) \ge 0$ be a nonnegative function which grows faster than x at infinity, *i.e.*, $\phi(x)/x \to \infty$ as $x \to \infty$. Let $\mathcal{C}$ be a collection of random variables such that, for some fixed $B < \infty$ and all $Z \in \mathcal{C}$,

$$\mathsf{E}\big(\phi(|Z|)\big) \le B.$$

Show that $\mathcal{C}$ is UI. In particular, any collection of random variables that is bounded uniformly in L_p for some $p > 1$ is also UI.

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[2] Although $\{X_i\}$ are UI, that won't be a factor in solving this problem. Is independence needed?