Sta 711: Homework 10

Conditional Expectation

1. Let $\{N_t\}_{t\geq 0}$ be a homogeneous Poisson process with rate λ, so $N_0 = 0$ and for every $n \in \mathbb{N}$ and $0 = t_0 < t_1 < ... < t_n < \infty$ the random variables $X_i := [N_{t_i} - N_{t_{i-1}}]$ for $1 \leq i \leq n$ are independent with marginal distributions $X_i \sim \mathsf{Po}\big(\lambda(t_i - t_{i-1})\big)$. For $0 < s < t < \infty$ find the conditional expectations:

   $\mathsf{E}[N_s \mid N_t] =$ $\qquad\qquad$ $\mathsf{E}[N_t \mid N_s] =$

2. Let $\{X_1, X_2\} \overset{\text{iid}}{\sim} \mathsf{Ex}(1)$ be iid unit-rate exponential random variables, and $t > 0$ a constant. Find:

   (a) $\mathsf{E}[X_1 \mid X_1 + X_2] =$

   (b) $\mathsf{P}[X_1 < 3 \mid X_1 + X_2] =$

   (c) $\mathsf{E}[X_1 \mid X_1 \wedge t] =$

   (d) $\mathsf{E}[X_1 \mid X_1 \vee t] =$

3. Let $X, Y \in L_2(\Omega, \mathcal{F}, \mathsf{P})$ and suppose $\mathsf{E}[X \mid Y] = \phi(Y)$ for a monotonically decreasing Borel function $\phi : \mathbb{R} \to \mathbb{R}$. Prove that $\mathsf{Cov}(X, Y) \leq 0$.

4. If $X \in L_2(\Omega, \mathcal{F}, \mathsf{P})$ and $\mathcal{H} \subset \mathcal{G} \subset \mathcal{F}$, prove[1]

   $$\mathsf{E}\Big[\big(X - \mathsf{E}[X \mid \mathcal{H}]\big)^2\Big] \geq \mathsf{E}\Big[\big(X - \mathsf{E}[X \mid \mathcal{G}]\big)^2\Big]$$

   In English, the MSE of the conditional expectation $\mathsf{E}[X \mid \mathcal{G}]$ given the bigger σ-algebra $\mathcal{G}$ is smaller than the MSE of the conditional expectation $\mathsf{E}[X \mid \mathcal{H}]$ given the smaller σ-algebra $\mathcal{H}$.

   What does this imply for the trivial σ-algebra $\mathcal{H} = \{\emptyset, \Omega\}$?

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[1] Don't just cite the class notes claim that $\mathsf{E}[X \mid \mathcal{G}]$ minimizes $\|X - Y\|_2$ over all $\mathcal{G}$-measurable Y, unless you prove that too.