Exercise 1

By Bayes' theorem, $p(\theta|x) \propto p(\theta)p(x|\theta)$ where the proportionality constant depends on x but not θ. Hence

$$\begin{aligned} O(1|x) =& p(1|x)/p(0|x) = \{p(1)p(x|1)\}/\{p(0)p(x|0)\} \\ =& \{0.01 \exp -(x-m_1)^2/2\tau^2\}/\{0.99 \exp -(x-m_0)^2/2\tau^2\} \\ =& 99^{-1} \exp -((x-m_1)^2 - (x-m_0)^2)/2\tau^2. \end{aligned}$$

So $O(1|x) > 99^{-1}k$ if, and only if,

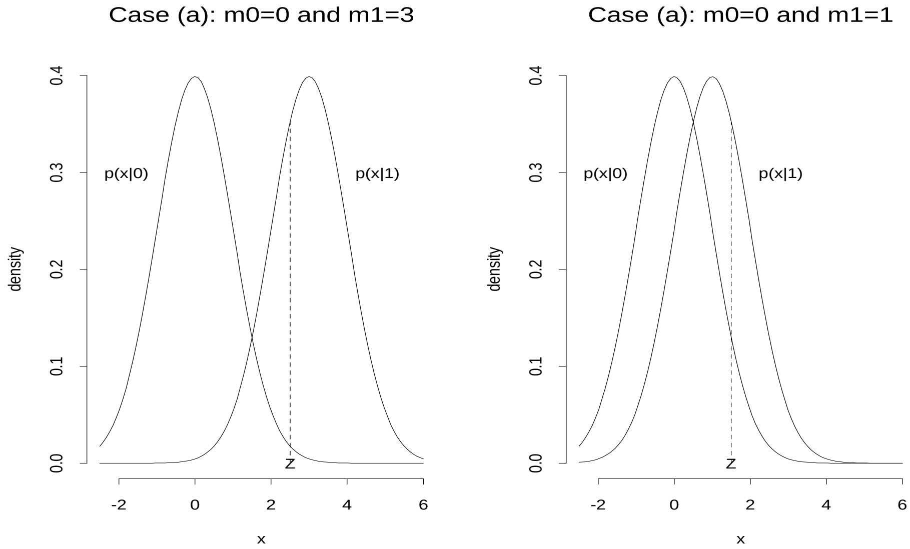
$$-((x-m_1)^2 - (x-m_0)^2)/2\tau^2 > \log(k).$$

Expanding the quadratics here, and using the identity $m_1^2 - m_0^2 = (m_1 - m_0)(m_1 + m_0)$, leads to

$$(x - \bar{m})(m_1 - m_0) > \log(k)\tau^2,$$

and the result follows as stated.

The amount by which the odds on disease increase following the test, as evidence in favour of disease, is going to be large when x is large: the story is clear if you graph the two densities as function of x, as below. Clearly $x > \bar{m}$ favours disease over health, but just how large is large depends on the ratio $(m_1 - m_0)/\tau^2$. For a fixed increase in odds k, larger values of x are needed for larger τ (since then the two populations overlap more if τ) and also for smaller $m_1 - m_0$ (since then the two populations are closer together, so harder to distinguish.) The factor $p(x|1)/p(x|0)$ is called the *likelihood ratio* or *Bayes' factor* for disease versus health provided by the datum x.



For x at a fixed distance above the intersection of the densities (i.e., the point Z), the ratio of densities $p(x|1)/p(x|0)$ is clearly larger in case (a), when the overlap is small, than in case (b).

Exercise 2

Write $z_i = 1$ of item i is truly defective, $z_i = 0$ otherwise. Then the z_i are latent variables, independent with $Pr(z_i = 1|\pi) = 1 - \pi$. Further, the chance of miss-classification means that $Pr(x_i = 1|z_i = 1, \pi) = Pr(x_i = 1|z_i = 1) = 0.99$ and $Pr(x_i = 1|z_i = 0, \pi) = Pr(x_i = 1|z_i = 0) = 0.01$.

(a) So

$$\begin{aligned} Pr(x_i = 1|\pi) &= Pr(x_i = 1|z_i = 1)Pr(z_i = 1|\pi) + Pr(x_i = 1|z_i = 0)Pr(z_i = 0|\pi) \\ &= 0.99(1 - \pi) + 0.01\pi. \end{aligned}$$

This reduces to $0.99 - 0.98\pi$. Similarly $Pr(x_i = 0|\pi) = 1-$ this or just $0.01 + 0.98\pi$.

(b) Bayes' theorem:

$$p(\pi|x_1) = k.p(\pi)p(x_1|\pi) = c.Pr(x_1|\pi)$$

for $0 < \pi < 1$. At $x = 1$ this gives $k(0.99 - 0.98\pi)$ and at $x = 0$ it gives $k.(0.01 + 0.98\pi)$. Here constants c and k ensure the densities integrate to one, so, by simple calculus, $k = c = 2$.

(c) The likelihood function from $x = \{x_1, \ldots, x_n\}$ is simply

$$p(x|\pi) = (0.99 - 0.98\pi)^t(0.01 + 0.98\pi)^{n-t}$$

where t is the number of defective items in the batch, $t = \sum_{i=1}^n x_i$. The uniform prior means that

$$p(\pi|x) = c.(0.99 - 0.98\pi)^t(0.01 + 0.98\pi)^{n-t}$$

for some constant c.

Exercise 3

No solution here. You have the answers.
