
Exercise 1

n.b. The problem uses the gamma density. If μ is gamma with parameters (a, b) it has density $p(\mu) = \{b^a/\Gamma(a)\}\mu^{a-1}\exp(-b\mu)$ for $\mu > 0$. So any function of μ of the form $h(\mu)\mu^{a-1}\exp(-b\mu)$ is equal to $h(\mu)\Gamma(a)b^{-a}p(\mu)$. This is most useful in finding the posterior in this problem, and the idea obviously extends to other forms of distribution. For instance, to find the integral $\int h\mu^{a-1}\exp(-b\mu)d\mu$ for some constant h, we simply note that it equals $h\Gamma(a)b^{-a}\int p(\mu)d\mu = h\Gamma(a)b^{-a}$. Easy integration, no work required as we just "recognise" the form of the integrand and use the fact that densities integrate to unity. Practice, practice, practice and remember these kinds of "tricks".

Now for the solution for the astronomer. By Bayes' theorem,

$$\begin{aligned} p(\mu|x_t) &\propto p(\mu)p(x_t|\mu) \\ &\propto \{\exp(-\mu/m)/m\}\ \{(1-\pi)(bt)^{x_t}\exp(-bt)/x_t! + \pi((b+\mu)t)^{x_t}\exp(-(b+\mu)t)/x_t!\} \\ &\propto \exp(-\mu/m)\ \{(1-\pi) + \pi(1+\mu/b)^{x_t}\exp(-\mu t)\} \\ &= k\{(1-\pi)\exp(-\mu/m) + \pi(1+\mu/b)^{x_t}\exp(-\mu(t+1/m))\} \end{aligned}$$

where common terms not involving μ are "absorbed" into the constant k.

(a) If $x_t = 0$,

$$\begin{aligned} p(\mu|0) &= k\{(1-\pi)\exp(-\mu/m) + \pi\exp(-\mu(t+1/m))\} \\ &= k\{(1-\pi)m\ \exp(-\mu/m)/m + [\pi/(t+1/m)]\ \exp(-\mu(t+1/m))(t+1/m)\} \\ &= k\{(1-\pi)m\ Ga(\mu|1, 1/m) + [\pi/(t+1/m)]\ Ga(\mu|1, (t+1/m))\} \end{aligned}$$

where $Ga(\cdot|a, b)$ is the Gamma density with parameters (a, b), and the special case $a = 1$ is exponential. Integrating over $0 < \mu < 1$, the result must be 1 (normalisation) so that $1 = k\{(1-\pi)m + \pi/(t+1/m)\}$ hence

$$k = 1/\{(1-\pi)m + \pi/(t+1/m)\}.$$

Writing $\pi^* = k\pi/(t+1/m)$, the posterior is

$$p(\mu|0) = (1-\pi^*)Ga(\mu|1, 1/m) + \pi^* Ga(\mu|1, (t+1/m)),$$

a mixture of two exponentials. It also follows that $\pi^* = \pi/\{\pi + (1-\pi)(1+mt)\}$.

(b) Note that π^* is the posterior probability on the second component, corresponding to $\mu > 0$ i.e., consistent with a supernova (or other) event having happened. Similarly $1-\pi^*$ is the posterior probability that $\mu = 0$ given $x_t = 0$. Easy calculations show that $\pi^* < \pi$ here; so observing no counts always increases the chance that only background radiation is being observed (naturally!).

(c) The binomial expansion gives

$$(1+\mu/b)^{x_t} = \sum_{i=0}^{x_t}\binom{x_t}{i}(\mu/b)^i$$

so that

$$
\begin{aligned}
p(\mu|x_t) &\propto (1-\pi)\exp(-\mu/m) + \pi \sum_{i=0}^{x_t} \binom{x_t}{i} b^{-i}\mu^i \exp(-\mu(t+1/m))\} \\
&\propto (1-\pi)\exp(-\mu/m) + \pi \sum_{i=0}^{x_t} a_i Ga(\mu|i+1,(t+1/m))\}
\end{aligned}
$$

where, using the full form of the gamma density functions in the sum,

$$
a_i = \binom{x_t}{i} b^{-i}(t+1/m)^{-(i+1)}\Gamma(i+1).
$$

Obviously this is a mixture of exponential and gamma distributions, with mixing weights in the sum proportional to πa_i.

Exercise 2

See my S-Plus code on the STA 214 homepage.

Exercise 3

No answers.
