
Exercise 1

Denote the sum of the 10 observations classified as having mean μ_0 by r_0, and the sum of the 5 with mean μ_1 by r_1.

(a) Given the $n = 50$ classification indicators z, the 50 "truly unclassified" are allocated into two groups corresponding to the two component exponentials. Let t_0 and s_0 be the number and sum of the x_i corresponding to the μ_0 component, i.e., $t_0 = \sum_{i=1}^{50}(1-z_i)$ and $s_0 = \sum_{i=1}^{50}(1-z_i)x_i$. Let t_1 and s_1 be computed similarly for the second group, i.e., the number and sum of the x_i corresponding to the μ_0 component, i.e., $t_1 = 50 - t_1$ and $s_1 = \sum_{i=1}^{50} z_i x_i$. Then we have two independent random sample problems. For the first group, the likelihood function is proportional to

$$\mu_0^{-(10+t_0)} \exp(-(r_0+s_0)/\mu_0).$$

With conditional prior $U(\mu_0|0,\mu_1)$ at the current "fixed" value of μ_1, this implies a conditional posterior for μ_0 proportional to the likelihood above over $(0,\mu_1)$. Thus $\phi_0 \equiv 1/\mu_0$ has a gamma distribution $G(\phi_0|9+t_0, r_0+s_0)$ truncated to $\phi_0 > 1/\mu_1$.

Similarly, $\phi_1 \equiv 1/\mu_1$ has a gamma distribution $G(\phi_1|4+t_1, r_1+s_1)$ truncated to $\phi_1 < 1/\mu_0$.

(b) Given z and the μ_j, the conditional posterior for π is simply the usual Beta, $Be(\pi|t_1+1, t_0+1)$. Note in particular that the pre-classified data provide no information about π, of course.

(c) Given π and the μ_j, the classification indicators z_i are conditionally independent with posteriors $\pi_i^* = Pr(z_i = 1|\mu_0, \mu_1, \pi, x)$ defined in odds terms by

$$\begin{aligned} O_i = \frac{1-\pi^*}{\pi^*} &= \frac{(1-\pi)}{\pi}\frac{Exp(x_i|1/\mu_0)}{Exp(x_i|1/\mu_1)} \\ &= \frac{(1-\pi)\mu_1}{\pi\mu_0}\exp(x_i(\mu_1^{-1}-\mu_0^{-1})) \end{aligned}$$

Then $\pi_i^* = 1/(1+O_i)$.

My S-Plus code implements Gibbs sampling with these conditionals.

Exercise 2

(a) If s is known then $x_t \sim Po(x_t|\mu)$ if $s > t$, and $x_t \sim Po(x_t|\theta)$ if $s \le t$. So,

$$p(x_t|\mu,\theta) = Pr(s>t)Po(x_t|\mu) + Pr(s \le t)Po(x_t|\theta)$$

assuming s is independent of (μ, θ). Clearly the mixture weights are $Pr(s > t) = (T-t)/T$ and $Pr(s \le t) = 1 - Pr(s > t) = t/T$.

(b) $p(x|\mu,\theta,s) = \{\prod_{t=1}^{s-1} Po(x_t|\mu)\} \times \{\prod_{t=s}^{T} Po(x_t|\theta)\}$. This can be written as

$$p(x|\mu,\theta,s) = K\ \mu^{u_s}\mathrm{e}^{-(s-1)\mu}\ \theta^{v_s}\mathrm{e}^{-(T-s+1)\theta}$$

where $u_s = \sum_{t=1}^{s-1} x_t$ and $v_s = \sum_{t=s}^{T} x_t$, and $K = \prod_{t=1}^{T} x_i!$.

(c) Combing the independent exponential priors on μ and θ with the result in (b) we have

$$\begin{aligned} p(x|s) = &\int p(x|\mu,\theta,s)p(\mu)p(\theta)\ d\mu\ d\theta \\ =&C\int_0^\infty \mu^{u_s}\mathrm{e}^{-(10+s-1)\mu}\ d\mu \int_0^\infty \theta^{v_s}\mathrm{e}^{-(50+T-s+1)\theta}\ d\theta \\ =&\frac{\Gamma(u_s+1)}{(9+s)^{u_s+1}} \times \frac{\Gamma(v_s+1)}{(T-s+51)^{v_s+1}} \end{aligned}$$

over $s = 1,\ldots,T$, and where C does not depend on s. As a result of the uniform prior on s, we then have $p(s|x)$ defined by $Pr(s|x) = c.p(x|s)$ where $c = 1/\sum_{s=1}^{T} p(x|s)$.

(d) Note that, given s and x, μ and θ are independent with conditional posteriors $Ga(\mu|u_s+1, 9+s)$ and $Ga(\theta|v_s+1, T-s+51)$. To sample the full posterior we can therefore simply sample s from $p(s|x)$, then each of μ and θ from the gamma posterior at that sampled value of s.
