Selected Problems from HW2 pg 17. DeGroot

(11)

Let's first show that the $\{B_i\}$ are disjoint.

Proof:
Consider an element x of $B_i \cap B_j$. Without loss of generality we can assume that $i < j$. $x \in B_i$ implies that $x \in A_1^c \cap A_2^c \cap \cdots \cap A_{i-1}^c \cap A_i$. $x \in B_j$ implies that $x \in A_1^c \cap A_2^c \cap \cdots \cap A_i^c \cap \cdots \cap A_{j-1}^c \cap A_j$. So we have $x \in A_i \cap A_i^c = \phi$. So $B_i \cap B_j \subset \phi$. That is $B_i \cap B_j = \phi$.

Now let's show that $\cup_{i=1}^{\infty} A_i = \cup_{i=1}^{\infty} B_i$.

Proof:
(Part 1 - inclusion) Suppose that $x \in \cup_{i=1}^{\infty} A_i$. Then $x \in A_i$ for some i. Suppose that s is the smallest such index. So $x \in A_i^c$ for $i = 1, 2, \ldots, s-1$ and $x \in A_s$. So $x \in A_1^c \cap A_2^c \cap \cdots \cap A_{s-1}^c \cap A_s = B_s$. Thus, $x \in \cup_{i=1}^{\infty} B_i$, and $\cup_{i=1}^{\infty} A_i \subset \cup_{i=1}^{\infty} B_i$.
(Part 2 - containment) Suppose that $x \in \cup_{i=1}^{\infty} B_i$. Then $x \in B_i$ for some i. So $x \in B_i = A_1^c \cap A_2^c \cap \cdots \cap A_{i-1}^c \cap A_i$ which implies that $x \in A_i$ and $x \in \cup_{i=1}^{\infty} A_i$. So $\cup_{i=1}^{\infty} A_i \supset \cup_{i=1}^{\infty} B_i$.

Now by axiom 3:

$$\sum_{i=1}^{\infty} \Pr(B_i) = \Pr(\cup_{i=1}^{\infty} B_i) = \Pr(\cup_{i=1}^{\infty} A_i).$$

And the first part of the problem follows from the same arguments above with axiom 3 replaced by Theorem 2.

(12)

To show that for any events $A_1, \ldots, A_n$, $\Pr(\cup_{i=1}^{n} A_i) \le \sum_{i=1}^{n} \Pr(A_i)$.

We'll use induction. For the case $n = 2$:

$\Pr(A_1 \cup A_2) = \Pr(A_1) + \Pr(A_2) - \Pr(A_1 \cap A_2) \leq \Pr(A_1) + \Pr(A_2)$.
Now we'll suppose that the conjecture is true for n and we'll prove it for n+1.

$$\begin{aligned}
\Pr(\cup_{i=1}^{n+1} A_i) &= \Pr((\cup_{i=1}^{n} A_i) \cup A_{n+1}) \\
&= \Pr(\cup_{i=1}^{n} A_i) + \Pr(A_{n+1}) - \Pr((\cup_{i=1}^{n} A_i) \cap A_{n+1}) \\
&\leq \Pr(\cup_{i=1}^{n} A_i) + \Pr(A_{n+1}) \\
&\leq \sum_{i=1}^{n+1} \Pr(A_i).
\end{aligned}$$