Sta 104 / Mth 135 - Midterm 1 and Key

Show enough work to make your thought processes clear. Set up all calculations thoroughly, but there is no need to completely reduce every calculation. Use the back if necessary.

1) Joe is tired of being single, and all the usual methods have not yielded any success. So Joe has determined to ask girls that he meets if they'll marry him until he finds one who says yes. Suppose that the proportion of girls who might be inclined to say yes to him is p (p might be very small), and also suppose that each girl's answer is independent of all others. What is the probability that Joe gets a yes on the n^{th} asking?

Answer: Our sample space is

$$S = \{\{\text{yes}\}, \{\text{not yes}, \text{yes}\}, \{\text{not yes}, \text{not yes}, \text{yes}\}, \ldots\}.$$

And

$$\begin{aligned} P(\{\text{yes}\}) &= p \text{ (as given)}, \\ P(\{\text{not yes}, \text{yes}\}) &= (1-p)p \text{ (independence)}, \\ P(\{\text{not yes}, \text{not yes}, \text{yes}\}) &= (1-p)^2 p, \end{aligned}$$

and so on. So the probability that Joe gets his first yes on the n^{th} asking is the same as the probability that Joe gets a not yes for each of the first $n-1$ questions and a yes on the n^{th} asking, and this equals $(1-p)^{n-1}p$. This is a special case of the negative binomial distribution and it is called the geometric distribution.

2) Prove that for any sets A, B, and C, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Answer: This can be found in your class notes.

3) Four people have sat down to play cards. Out of the standard deck of 52 cards 13 are dealt to each person. What is the probability that one person will hold all the spades?

Answer: The first thing to note is that

$$\begin{aligned} P(\text{one person holds all of the spades}) =& P(\text{N holds all the spades}) \\ &+ P(\text{S holds all the spades}) \\ &+ P(\text{E holds all the spades}) \\ &+ P(\text{W holds all the spades}) \\ =& 4 * P(\text{N holds all the spades}). \end{aligned}$$

These probabilities add because the events are disjoint.

$$P(\text{N holds all the spades}) = \frac{\frac{13!}{13!0!0!0!}\frac{39!}{0!13!13!13!}}{\frac{52!}{(13!)^4}}.$$

So

$$P(\text{one person holds all of the spades}) = \frac{4 \cdot 13!39!}{52!}.$$

Which is not coincidentally

$$\frac{4}{C_{52,13}}.$$

This question is just like an example in the multinomial section of our textbook.

4) Suppose your sample space is the outcome of one roll of a balanced die. Then $S = \{1, 2, 3, 4, 5, 6\}$. Find subsets A and B of S such that:

a) A and B are not disjoint and not independent.

An answer: Take $A = \{1, 2\}$, and $B = \{1\}$. $A \cap B = \{1\}$ and so A and B are not disjoint. $P(A) = 1/3$ and $P(B) = 1/6$, but $P(A \cap B) = 1/6 \neq 1/6 \cdot 1/3 = P(A)P(B)$. So A and B are not independent.

b) A and B are disjoint but not independent.

An answer: Take $A = \{1, 3, 5\}$, and $B = \{2, 4, 6\}$. $A \cap B = \phi$ and so A and B are disjoint. $P(A) = 1/2$ and $P(B) = 1/2$, but $P(A \cap B) = 0 \neq 1/2 \cdot 1/2 = P(A)P(B)$. So A and B are not independent.

5) Suppose that you and your nemesis are of equal skill at the game of Go. Having a whole Saturday with nothing better to do you all set out to play six games. What is the probability that you will win more games than he does? (i.e. What is the probability that he buys the beer?) Assume, of course, that the outcomes of the individual games are independent and also that the probability of a draw on any individual game is zero.

Answer: Postponed.

6) Prove the most elemental form of Bayes' theorem: $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$.

Answer: In the book.

7) Prove for any events A and B that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Answer: In the book.

8) Prove that for events A and B, $A \subset B$ implies $P(A) \leq P(B)$.

Answer: In the book.

9) Four hundred people are at a party. What is the probability that they all have different birthdays?

Answer: If there were 366 people at the party they could all have different birthdays, but as soon as a 367^{th} shows up at least two people must share a birthday. So the probability that 400 people all have different birthdays is zero.

10) It is well known that vegetarians are at much less risk of contracting colon cancer than are those who eat animal flesh. Most studies suggest that the probability of contracting colon cancer for non-vegetarians is 3 times that for vegetarians. Suppose that the proportion of the American population that is vegetarian is v. If you learn that a certain individual has been diagnosed with colon cancer then what is the posterior probability that this individual is a vegetarian?

Answer: This is a straightforward application of Bayes' Theorem. The answer reduces to

$$\frac{v}{3-2v}.$$

The only trick is to note that $P(\text{colon cancer}|\text{non}-\text{vegetarian}) = 3 * P(\text{colon cancer}|\text{vegetarian})$