Chapter 13

Bayes Theorem and Bayesian Inference

Recall that multiplication rule claims:

$$P(AH) = P(A)P(H|A) = P(H)P(A|H).$$

This simple identity is the essence of **Bayes Theorem:**

$$P(H|A) = \frac{P(A|H)P(H)}{P(A)}.$$

Assume that the whole sample space S is partitioned by events $H_1, H_2, \ldots, H_n$. Events H_i we will cal **hypotheses** and they satisfy:

$$\begin{aligned} H_1 \cup H_2 \cup \ldots \cup H_n &= S \\ H_i \cap H_j &= \emptyset, i \neq j. \end{aligned}$$

If we are interested in probability of event A, then **formula of total probability** gives

$$P(A) = P(A|H_1)P(H_1) + P(A|H_2)P(H_2) + \ldots + P(A|H_n)P(H_n).$$

1. A new test has been devised for detecting a particular type of cancer. If the test is applied to the person who actually has that type of cancer, the probability that person will have a positive reaction is 0.995. If applied to a person who does not have this type of cancer, the probability that the person will have a (false) positive reaction is 0.012.

Suppose that test is given to patients at high risk of having cancer, and that one person, out of 50 in this group actually has this type of cancer.

(i) What is the probability that the randomly selected person from the group tests *positive?*

(ii) If randomly selected person tests *positive*, what is the probability that he/she **does not have** this type of cancer.

2. Jeremy, an enthusiastic Duke student, poses a statistical model for his scores on standard IQ tests. He thinks that, in general, his scores are normally distributed with unknown mean θ and the variance 80. Prior (and expert) opinion is that the true IQ of Duke students, θ, is a normal random variable, with mean 110 and the variance 120. Jeremy took the test and scored 98.

Estimate his true IQ θ in a Bayesian manner.

9. A student answers a multiple choice examination question that has 4 possible answers. Suppose that the probability that the student knows the answer to the question is 0.80 and the probability that the student guesses is 0.20. If student guesses, probability of correct answer is 0.25.

(i) What is the probability that the fixed question is answered correctly?

(ii) If it is answered correctly what is the probability that the student really knew the correct answer.

10. Factory has three types of machines producing an item. Probabilities that the item is I quality f it is produced on i-th machine are given in the following table:

machine	probability of I quality
1	0.8
2	0.7
3	0.9

The total production is done 30% on type I machine, 50% on type II, and 20% on type III.
One item is selected at random from the production.
(i) What is the probability that it is of I quality?
(ii) If it is of first quality, what is the probability that it was produced on the machine I?

11. One out of 1000 coins has two tails. The coin is selected at random out of these 1000 coins and flipped 5 times. If tails appeared all 5 times, what is the probability that the selected coin was 'two-tailed'?

19. In the city Kokomo, IN, 50% are conservatives, 30% are liberals and 20% are independents.

Records show that in a particular election 82% of conservatives voted, 65% of liberals voted and 50% of independents voted.

If the person from the city is selected at random and it is learned that he/she did not vote, what is the probability that the person is liberal?

1.[10 pts] A new test has been devised for detecting a particular type of cancer. If the test is applied to the person who actually has that type of cancer, the probability that person will have a positive reaction is 0.995. If applied to a person who does not have this type of cancer, the probability that the person will have a (false) positive reaction is 0.012.

Suppose that test is given to patients at high risk of having cancer, and that one person, out of 50 in this group actually has this type of cancer.

(i) What is the probability that the randomly selected person from the group tests *positive?*

(ii) If randomly selected person tests *positive*, what is the probability that he/she **does not have** this type of cancer.

A federalist paper resolved! Suppose that a work, the author of which is known to be either Madson or Hamilton, contains a certain key phrase. Suppose further that this phrase occurs in 60% of the papers known to have been written by Madison, but in only 20% of those by Hamilton. Finally, suppose a historian gives subjective probability .3 to the event that the author is Madison (and consequently .7 to the complementary event that the author is Hamilton). Compute the historian's posterior probability for the event that the author is Madison.

Transylvania. In a small village in Transylvania 15% of the population are vampires, 20% are ghosts, and 65% are ordinary people. Vampires never tell the truth, ghosts tell the truth 37% of the time, and ordinary people tell the truth 95% of the time. It is impossible to tell apart vampires, ghosts and ordinary people by the way they look (between 6:00am and 11:59:59pm). You are introduced to a gentleman from the village.

(a) What is the probability that he did not give you his real name?

(b) Just at 11:59:59pm you realize that the gentleman lied about his name. What is the probability that your companion is a vampire or ghost?

1. Robin Hood had finally been caught by the sheriff of Nottingham and was scheduled for execution. Because Robin was a popular her and the sheriff wanted to seem generous, he offered Robin a chance to go free: "Here is box with 8 black balls and 2 white balls. You will pick one ball from the box while blindfolded, and if you pick a white ball you go free. If you pick a black ball, you die." Robin, who had taken a statistics course, proposed instead that he (Robin) be allowed to sort the balls into two boxes. The sheriff would then choose one of the two boxes at random and Robin would select from that box while blindfolded. The sheriff thought that Robin's suggestion would make no difference (the odds that Robin would go free would still be only $\frac{2}{10} = 0.20$). Hence he agreed to

Robin's proposal. The sheriff should have taken a statistics course. Below are three ways in which Robin could arrange the 8 black balls and 2 white balls in the two boxes. In each case, find the probability that Robin goes free (picks a white ball).
(a) $| \circ \circ || \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet |$.
(b) $| \circ \bullet || \circ \bullet \bullet \bullet \bullet \bullet \bullet \bullet |$.
(c) $| \circ || \circ \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet |$.

College Entrance Test. Because of the role of college aptitude test scores in college entrance decision, there are minicourses that purport to teach students how to take these tests. A particular aptitude test has been found to produce scores that are normally distributed, with mean θ and standard deviation 80. If the minicourse directed at this test is effective (on average), the mean score θ of students who take the course is larger than 500; otherwise it is not. We want to test

$$H_0 : \theta \leq 500 \quad \text{versus} \quad H_1 : \theta > 500,$$

and our prior for θ is $N(520, 20^2)$.

1. Find the prior probabilities of hypotheses H_0 and H_1, π_0 and π_1.

2. If 50 observations gives the mean 513, find the posterior probabilities of hypotheses H_0 and H_1, p_0 and p_1, and make the decision.

3. What is 90% credible set for θ.

Solution: XXXXXXXXXXX)

Two Masked Robbers. Two masked robbers try to rob a crowded bank during the lunch hour but the teller presses a button that sets off an alarm and locks the front door. The robbers realizing they are trapped, throw away their masks and disappear into the chaotic crowd. Confronted with 40 people claiming they are innocent, the police gives everyone a lie detector test. Suppose that guilty people are detected with probability 0.85 and innocent people appear to be guilty with probability 0.08. What is the probability that Mr. Smith was one of the robbers given that the lie detector says he is?

Guessing Subject in an experiment are told that either a red or a green light will flash. Each subject is to guess which light will flash. The subject is told that the probability of a red light is 0.7, independent of guesses. Assume that the subject is a probability matcher- that is , guesses red with probability .70 and green with probability .30.

(a) What is the probability that the subject guesses correctly?

(b) Given that a subject guesses correctly, what is the probability that the light flashed red?

Toothpaste. A research study on fluoride in toothpaste was conducted using Colgate's *MFP* formula, and a leading stannous fluoride (*SF*) toothpaste. Data[1] on the number of new cavities over three year period are summarized as follows:

	n	Mean	s.d.
MFP	208	19.98	10.60
SF	201	22.39	11.96

The mean difference, X, in number of new cavities between the *MFP* and *SF* samples is modeled by $N(\theta, 1.12^2)$ The null hypothesis is $\theta \leq 0$. Assume that the prior for θ is $N(4, 2^2)$.

(1) Find the posterior distribution for θ given the observed mean difference $X = 2.41$.

(2) Test the hypothesis H_0.

(3) Find 95% credible set for θ. Compare the credible set with the 95% confidence interval.

Bayes and bats. By careful examination of sound and film records it is possible to measure the distance at which a bat first detects an insect. The measurements are modeled by normal distribution $N(\theta, 10^2)$, where θ is the unknown mean distance (in cm).

Experts believe that the prior suitably expressing uncertainty about θ is $\theta \sim N(50, 10^2)$.

Three measurements are obtained: 62, 52, and 68.

(a) Find the posterior distribution of θ given the observations.

(b) Test the hypothesis H_0 that $\theta \geq 50$ in Bayesian fashion.

[1]Frankl, S. and Alman, J. *J. Oral. Therapeutics Pharmacol. 4 (1968), 443-449.*

(c) What is the 95% credible set for θ.

Crystal. Chrystal (1891) wrote: *No one would say that, if you simply put two white balls in a bag containing one of unknown color, equally likely to be white or black, that this action raised the odds that the unknown ball is white ftro even to 3:1.* If we draw the 3 balls and the two first are white, is Chrystal's argument still valid? (Note: He used this argument to reject the Bayes rule.)

College Entrance Test Again. Because of the role of college aptitude test scores in college entrance decision, there are minicourses that purport to teach students how to take these tests. A particular aptitude test has been found to produce scores that are normally distributed, with mean θ and standard deviation 60. If the minicourse directed at this test is effective (on average), the mean score θ of students who take the course is larger than 500; otherwise it is not. We want to test

$$H_0 : 480 \leq \theta \leq 520 \quad \text{versus} \quad H_1 : \theta \text{ not in } [480, 520],$$

and our prior for θ is $N(520, 30^2)$.

- (i) If 25 observations gives the mean 510, perform the test and make the decision.
- (ii) Find 96% credible set for θ. Compare the obtained credible set with the frequentist 96% confidence interval for the unknown mean θ. Explain why credible sets tend to be shorter than the corresponding confidence intervals.

Weapons at Airports. A recent test conducted by the Federal Aviation Administration found that guards hired to screen passengers for weapons at the boarding gate had a very poor rate of detection. The detection rates for weapons carried by F.A.A. inspectors or placed in their carry-on-luggage averaged 80%, but their rates varied from 34% to 99% for the airports tested (*New York Times,* June 18, 1987). Suppose that in a particular city, airport **A** handles 50% of all airline traffic, while airports **B** and **C** handle 30% and 20%, respectively. The detection rates at the three airports are 0.9, 0.5, and 0.4, respectively. If the passenger at one of the airports is found to be carrying a weapon through the boarding gate, what is the probability that the passenger is not using the airport **C**?