Bayes Statistics Handout

Estimation. Model: $X|\theta \sim N(\theta, \sigma^2)$.
Prior: $\theta \sim N(\mu, \tau^2)$
Posterior: $\theta|X \sim N(\frac{\tau^2}{\sigma^2+\tau^2}X + \frac{\sigma^2}{\sigma^2+\tau^2}\mu, \frac{\tau^2\sigma^2}{\sigma^2+\tau^2})$.

Intelligence Test. A child is given an intelligence test. The results of the test can be modeled by: $X|\theta \sim N(\theta, 10^2)$, where θ is the unknown IQ. From the past experience and an expert opinion the prior $\theta \sim N(100, 15^2)$ was posed. The child scored X=115. Estimate θ in Bayesian fashion.
Solution. The posterior is:

$$\theta|X \sim N(\frac{\tau^2}{\sigma^2+\tau^2}X + \frac{\sigma^2}{\sigma^2+\tau^2}\mu, \frac{\tau^2\sigma^2}{\sigma^2+\tau^2})$$

The mean of the posterior is the Bayes estimate:

$$\hat{\theta} = \frac{\tau^2}{\sigma^2+\tau^2}X + \frac{\sigma^2}{\sigma^2+\tau^2}\mu = \frac{225}{325}\cdot 115 + \frac{100}{325}\cdot 100 = 110.3846$$

The posterior is $N(110.3846, 8.32^2)$. **Testing**

Test $H_0 : \theta \in \Theta_0$ versus $H_1 : \theta \in \Theta_1$.
Prior Probabilities: $\pi_0 = P^{pr}(\Theta_0)$; $\pi_1 = P^{pr}(\Theta_1)$.
Posterior Probabilities: $p_0 = P^{po}(\Theta_0)$; $p_1 = P^{po}(\Theta_1)$.
Decide the hypothesis with larger posterior probability.

Example: Let $X|\theta \sim N(\theta, 1)$ and $\theta \sim N(2, 2^2)$. We want to test $H_0 : \theta \le 2$ vs $H_1 : \theta > 2$, when the sample $X_1 = 2, X_2 = 4, X_3 = 3$, and $X_4 = 2$ is observed.
First we find prior probabilities of hypotheses.

$$\pi_0 = P^{pr}(\Theta_0) = P^{pr}(\theta \le 2) = P(Z \le \frac{2-2}{2}) = P(Z \le 0) = \frac{1}{2}.$$

Consequently, $\pi_1 = P^{pr}(\Theta_1) = 0.5$.
Since $X|\theta \sim N(\theta, 1)$ then $\bar{X} \sim N(\theta, 1/n)$. In our case, $\bar{X} \sim N(\theta, (1/2)^2)$ Since $\bar{X} = \frac{2+4+3+2}{4} = 2.75 was observed$, the posterior is:

$$\theta|\bar{X} \sim N(\frac{1/4}{1/4+4}2 + \frac{4}{1/4+4}2.75, \frac{4\cdot 1/4}{1/4+4}) = N(2.7059, 0.4851^2)$$

Now, $p_0 = P^{po}(\Theta_0) = P^{po}(\theta \le 2) = P(Z \le \frac{2-2.7059}{0.4851}) = P(Z \le -1.46) = 0.5 - \Phi(1.46) = 0.5 - 0.4279 = 0.07$.

Since $P^{po}(\Theta_0) = 0.07 < P^{po}(\Theta_1) = 0.93$, decide H_1.

Credible Sets. The $(1-\alpha)100\%$ credible set is the smallest set (interval) whose posterior probability is $(1-\alpha)$.

Example: In the **Intelligence Test** example find the 95% credible set for the unknown IQ θ.

Solution: The posterior is: $N(110.3846, 8.3205^2)$ The desired credible set is $(110.3846 - \delta, 110.3846 + \delta)$ where

$$P^{po}(110.3846 - \delta \leq X \leq 110.3846 + \delta) = 0.95.$$

After standardization we get $P(-\frac{\delta}{8.32} \leq Z \leq \frac{\delta}{8.32}) = 0.95$.
$\Phi(\frac{\delta}{8.32}) = 0.475$,
$\frac{\delta}{8.32} = 1.96$.
$\delta = 16.3072$.
Thus, the 95% credible set is:

$$[110.3846 - 16.3072, 110.3846 + 16.3072] = [94.0774, 126.6918].$$

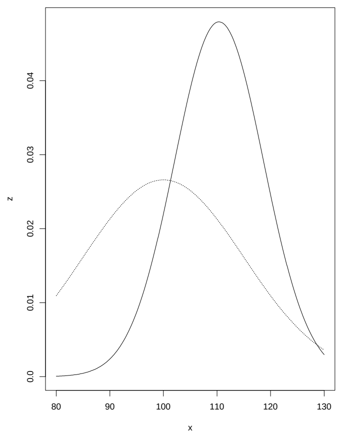

Figure 1: Prior and posterior densities for the **Intelligence** example.