STA213 **Fall 1999**

Answers to Midterm Exam

1. (a) First, note that the conditional distribution of X, given $Y = y$ (i.e., $X|Y = y$) is $N(\rho y, 1-\rho^2)$. Hence,

$$\begin{aligned} E(X^2|Y=y) &= Var(X|Y=y) + (E(X|Y=y))^2 \\ &= 1-\rho^2+\rho^2y^2. \end{aligned}$$

(b) To compute $Corr(X^2, Y)$, we need to know $E(X^2Y)$ and $Var(X^2)$.

$$\begin{aligned} E(X^2Y) &= E[YE(X^2|Y)] \\ &= E[Y(1-\rho^2+\rho^2Y^2)] \text{ from part (a)} \\ &= 0. \end{aligned}$$

The moment generating function of X is $e^{t^2/2}$. By differentiating 4 times and plugging in $t = 0$, we can get $E(X^4) = 3$.

$$\begin{aligned} Var(X^2) &= E(X^4) - (E(X^2))^2 \\ &= 3-1 = 2. \end{aligned}$$

Combining the above results,

$$\begin{aligned} Corr(X^2, Y) &= \frac{Cov(X^2, Y)}{\sqrt{Var(X^2)}\sqrt{Var(Y)}} \\ &= \frac{E(X^2Y) - E(X^2)E(Y)}{\sqrt{2}} \\ &= 0. \end{aligned}$$

2. The number of customers N follows $Poi(\lambda)$ with $\lambda = 40$. Let S be Dave's revenue for a day. If $N = n > 0$, then $S = \sum_{i=1}^{n} X_i$, where X_i follows iid $Exp(\mu)$ with $\mu = 1/20$. If $N = 0$, then $S = 0$. We need to compute $E(S)$ and $Var(S)$. Note first $E(X_i) = 1/\mu$ and $Var(X_i) = 1/\mu^2$ for all $i = 1, 2, \cdots$. If $N = 0$ (no customers), $E(S|N = 0) = 0$ and $Var(S|N = 0) = 0$. If $N = n > 0$, then

$$\begin{aligned} E(S|N=n) &= E(X_1+\cdots+X_n|N=n) \\ &= E(X_1+\cdots+X_n) = \frac{n}{\mu} \end{aligned}$$

and

$$\begin{aligned} Var(S|N=n) &= Var(X_1+\cdots+X_n) \\ &= \frac{n}{\mu^2}. \end{aligned}$$

In summary, we have $E(S|N)=N/\mu$ and $Var(S|N)=N/\mu^2$. Because $E(N)=\lambda$,

$$E(S)=E[E(S|N)]=E[N/\mu]=\frac{\lambda}{\mu}=40\cdot 20=\$800.$$

To compute $Var(S)$, we use Eve's formular.

$$\begin{aligned} Var(S) &= E(Var(S|N))+Var(E(S|N)) \\ &= E(\frac{N}{\mu^2})+Var(\frac{N}{\mu}) \\ &= \frac{\lambda}{\mu^2}+\frac{\lambda}{\mu^2} \\ &= \frac{2\lambda}{\mu^2} \\ &= 2\cdot 40\cdot 20^2=320000. \end{aligned}$$

3. We need to compute

$$prob=P(x_1+\zeta_1\geq x_i+\zeta_i,\ \text{for all } i=2,\cdots k).$$

First, note that $Z=e^{-\zeta_1}\sim Exp(1)$ and, for any $a>0$,

$$E(e^{-aZ})=\int_0^\infty e^{-(a+1)x}dx=\frac{1}{1+a}.$$

$$\begin{aligned} prob &= E[P(x_1-x_i+\zeta_1\geq\zeta_i,\ \text{for all } i=2,\cdots,k|\zeta_1)] \\ &= E(\prod_{i=2}^n P(\zeta_i\leq x_1-x_i+\zeta_1|\zeta_1)) \\ &= E(\prod_{i=2}^n \exp(-e^{-(x_1-x_i+\zeta_1)})) \\ &= E(\exp(-e^{-\zeta_1}\sum_{i=2}^n e^{x_i-x_1})) \end{aligned}$$

$$
\begin{aligned}
&= E(\exp(-Z\sum_{i=2}^{k} e^{x_i - x_1})) \\
&= \frac{1}{1+\sum_{i=2}^{n} e^{x_i-x_1}} \\
&= \frac{e^{x_1}}{\sum_{i=1}^{n} e^{x_i}}.
\end{aligned}
$$

The conclusion of this problem is a basis of multinomial logistic model.

4. ($\Leftarrow$) We need to show $Corr(X,Y) = Corr(X, aX+b) = \pm 1$. Note

$$Cov(X,Y) = Cov(X, aX+b) = aVar(X)$$

and

$$Var(Y) = Var(aX+b) = a^2 Var(X).$$

Since $Var(Y) > 0$, $a \neq 0$.

$$
\begin{aligned}
Corr(X,Y) &= \frac{Cov(X,Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}} \\
&= \frac{aVar(X)}{|a|Var(X)} \\
&= \begin{pmatrix} 1, & if a > 0 \\ -1, & if a < 0. \end{pmatrix}
\end{aligned}
$$

($\Rightarrow$) It suffices to show

$$P(\frac{Y-E(Y)}{\sqrt{Var(Y)}} = \rho\frac{X-E(X)}{\sqrt{Var(X)}}) = 1.$$

Let

$$
\begin{aligned}
U &= \frac{Y-E(Y)}{\sqrt{Var(Y)}} \\
V &= \frac{X-E(X)}{\sqrt{Var(X)}}.
\end{aligned}
$$

We need to show $P(U = \rho V) = 1$. Note $E(U - \rho V) = 0$ and

$$Var(U - \rho V) = 1 - \rho^2 = 0,$$

because $\rho = \pm 1$. Since the random variable $U - \rho V$ has mean 0 and variance 0, by a property of variance,

$$P(U = \rho V) = P(U - \rho V = 0) = 1.$$

5. Suppose the specific customer we consider is J. Define a random variable

$$V = \left(\begin{array}{ll} 1, & \text{if J wins} \\ 0, & \text{otherwise.} \end{array} \right.$$

The amount of winning prize of J is, then,

$$W = \frac{P}{K} V I(K > 0).$$

We need to compute the expectation of W. The probability that a customer guesses all the six numbers correctly is

$$r = \frac{1}{\binom{10}{6}} = \frac{1}{210};$$

hence, the distribution of K is binomial distribution with parameter N and r, i.e. $K \sim Bin(N, r)$. Suppose there are K winning customers of the game. Then the probability that J is one of them is K/N; hence,

$$P(V = 1|K) = \frac{K}{N}.$$

Now, we are in position to compute $E(W)$.

$$\begin{aligned}
E(W) &= E[E(\frac{P}{K} V I(K > 0)|K)] \\
&= E[\frac{P}{K} I(K > 0) E(V|K)] \\
&= E[\frac{P}{K} I(K > 0) \frac{K}{N}] \\
&= E[\frac{P}{N} I(K > 0)] \\
&= \frac{P}{N} P(K > 0) \\
&= \frac{P}{N}(1 - (1 - r)^N).
\end{aligned}$$

6. Note that the transformation $T : (X, Y) \to (R, A)$ is one to one and onto function from R^2 to $[0, \infty) \times [0, 2\pi)$. The inverse of T is the familiar polar coordinate transformation;

$$T^{-1} : \begin{pmatrix} R \\ A \end{pmatrix} \longrightarrow \begin{pmatrix} X = R\cos(A) \\ Y = R\sin(A) \end{pmatrix}.$$

The jacobian of T^{-1} is

$$||J|| = \left|\left| \begin{matrix} \cos(A) & -R\sin(A) \\ \sin(A) & R\cos(A) \end{matrix} \right|\right| = R.$$

Since the density of (X, Y) is

$$\frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}},$$

the density of (R, A) is

$$\frac{1}{2\pi} e^{-\frac{r^2}{2}} r I(r > 0) I(0 \leq a < 2\pi).$$

Hence, the R and A are independent; $A \sim Unif(0, 2\pi)$; R has the density $re^{-r^2/2}$. Again, using the transformation formular, the density of $Z = R^2$ is

$$\frac{1}{2} e^{-\frac{z}{2}},$$

which is $Exp(1/2) = Gamma(1, 1/2) = \chi^2_2$.