STA213 **Fall 1999**

Homework 2

Suggested problems :

p101 : 1,4,5,8,10 p107 : 2,3,4,7,8,10 p113 : 1,3,4,5,7 p124 : 1,2,4,5,6 p132 : 1,2,3,6,7,8 p139 : 1,2,6,7,8,9 p149 : 1,2,3 p157 : 1,2,6,7 p169 : 1,2,3,4,5,6,7,13,15

Problems for grading :

1. Let X, Y be iid $Exp(1)$.

(a) Find the pdf of rv's $X+Y$, $X-Y$, XY, X/Y, $\min\{X,Y\}$, $\max\{X,Y\}$, $\min\{X,Y\}/\max\{X,Y\}$,$X/(X+Y)$.

(b) Let $U = X+Y$ and $V = X-Y$. Find the conditional pdf of V, given $U = u$, for some fixed $u > 0$.

(c) Show that U and $Z = X/(X+Y)$ are independent.

2. (Box and Muller Transformation)
A random variable X has the normal distribution, if it has a pdf

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}, \text{ for } x \in R,$$

where $\mu \in R$ and $\sigma > 0$; we denote $X \sim N(\mu, \sigma^2)$. Let U_1, U_2 be iid $Unif(0,1)$. Define

$$\begin{aligned} X_1 &= (-2\log U_1)^{1/2}\cos(2\pi U_2) \\ X_2 &= (-2\log U_1)^{1/2}\sin(2\pi U_2). \end{aligned}$$

Show X_1, X_2 are iid $N(0,1)$.

3. (Bivariate Normal Distribution)
A two-dimensional rv (X_1, X_2) is said to have a bivariate normal distribution if the joint pdf is of the form, for $x_1, x_2 \in R$,

$$f(x_1,x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp[-\frac{1}{2(1-\rho^2)}\{(\frac{x_1-\mu_1}{\sigma_1})^2 - 2\rho(\frac{x_1-\mu_1}{\sigma_1})(\frac{x_2-\mu_2}{\sigma_2}) + (\frac{x_2-\mu_2}{\sigma_2})^2\}],$$

where $\sigma_1 > 0$, $\sigma_2 > 0$, $|\rho| < 1$ and $\mu_1, \mu_2 \in R$.

(a) Let $Z_1 = (X_1 - \mu_1)/\sigma_1$ and $Z_2 = (X_2 - \mu_2)/\sigma_2$. Find the joint pdf of (Z_1, Z_2).

(b) Show that marginally $X_1 \sim N(\mu_1, \sigma_1^2)$ and $X_2 \sim N(\mu_2, \sigma_2^2)$.

(c) Show that the conditional distribution of X_1, given $X_2 = x_2$ is

$$N(\mu_1 + \rho\frac{\sigma_1}{\sigma_2}(x_2 - \mu_2), \sigma_1^2(1 - \rho^2)).$$

4. We say a discrete random variable has a binomial distribution with parameter n and p, if it has pf

$$P(X = x) = \binom{n}{x} p^x (1 - p)^{n-x}, \text{ for } x = 0, 1, 2, \cdots, n;$$

we denote $X \sim Bin(n, p)$. Suppose X, Y are iid with $Bin(n, p)$.

(a) Find the pf of $X + Y$.

(b) Find the joint pf of $(U = X/(Y + 1), V = Y + 1)$.

5. Let f_1, f_2, f_3 be three pdfs with corresponding cdfs F_1, F_2, F_3, and let α be a constant with $|\alpha| \leq 1$. Define

$$f_\alpha(x_1, x_2, x_3) = f_1(x_1)f_2(x_2)f_3(x_3)[1 + \alpha(2F_1(x_1) - 1)(2F_2(x_2) - 1)(2F_3(x_3) - 1)].$$

(a) Show that f_α is a pdf for each $\alpha \in [-1, 1]$.

(b) Show that, for each $\alpha \in [-1, 1]$, f_α has marginal densities f_1, f_2, f_3.

6. (Structure of exponential order statistics)
Let $X_{(1)}, \cdots, X_{(n)}$ be the set of order statistics of independent rv's $X_1, \cdots, X_n$ having $Exp(\beta)$, i.e. the pdf of X_i is

$$f(x) = \beta e^{-\beta x} I(x > 0).$$

(a) Show that $X_{(r)}$ and $X_{(s)} - X_{(r)}$ are independent for any $s > r$.

(b) Find the pdf of $X_{(r+1)} - X_{(r)}$.

(c) Let $Z_1 = nX_{(1)}, Z_2 = (n - 1)(X_{(2)} - X_{(1)}), Z_3 = (n - 2)(X_{(3)} - X_{(2)}), \cdots, Z_n = (X_{(n)} - X_{(n-1)})$. Show that $(Z_1, Z_2, \cdots, Z_n)$ and $(X_1, X_2, \cdots, X_n)$ have the same joint distribution.