STA213 **Fall 1999**

Homework 3

Suggested problems :

p186 : 1,2,3,7,8 p193 : 1,2,8,9 p199 : 6 p205 : 1,2,4,8,9,10
p218 : 2,9,11,12,13 p224 : 5,6 p231 : 3,8

Problems for grading :

1. Let X be a random variable. Suppose all the relavent quantities exist. Show that

$$\begin{aligned} E(X) &= \int_0^\infty P(X > x)dx - \int_{-\infty}^0 P(X \le x)dx \\ E|X|^\alpha &= \alpha \int_0^\infty x^{\alpha-1} P(|X| > x)dx, \text{ for } \alpha > 0. \end{aligned}$$

2. (Gamma Function)
Let

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx.$$

Show the following properties.

(a) $\Gamma(\alpha) < \infty$, for $\alpha > 0$.

(b) $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$, for $\alpha > 1$.

(c) $\Gamma(n) = (n-1)!$, for all positive interger n.

(d)

$$\int_0^\infty x^{\alpha-1} e^{-\beta x} dx = \frac{\Gamma(\alpha)}{\beta^\alpha},$$

for $\alpha, \beta > 0$. Note that this shows that gamma density is indeed a pdf.

(e)

$$\int_{-\infty}^\infty e^{-\frac{1}{2}x^2} dx = \sqrt{2\pi}.$$

(Hint : first, use the polar coordinate transformation to show

$$\int_{-\infty}^\infty \int_{-\infty}^\infty e^{-\frac{1}{2}(x^2+y^2)} dx dy = 2\pi.$$

Note that this shows that a normal density is indeed a normal density.)

(f) $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

(g)

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \text{ for } \alpha, \beta > 0.$$

Note this shows that a beta density is indeed a density.

(h)

$$\int \cdots \int_{x_i>0, \sum_{i=1}^{k-1} x_i<1} \prod_{i=1}^{k-1} x_i^{\alpha_i-1}(1-\sum_{i=1}^{k-1} x_i)^{\alpha_k-1}dx_1\cdots dx_{k-1} = \frac{\prod_{i=1}^k \Gamma(\alpha_i)}{\Gamma(\sum_{i=1}^k \alpha_i)},$$

for $\alpha_i > 0$, $i = 1, \cdots, k$. Note this shows a Dirichlet density is indeed a density.

3. (Lyapunov Inequality)
Suppose $E|X|^s < \infty$ and $1 < r \le s < \infty$. Recall $||X||_p = (E(|X|^p))^{1/p}$.
Show $||X||_r \le ||X||_s$. (Hint : use Hölder inequality.)

4. Let X be a random variable with mgf $\phi(t)$, $-h < t < h$. Prove that

$$\begin{aligned} P(X \ge a) &\le e^{-at}\phi(t), 0 < t < h, \\ P(X \le a) &\le e^{-at}\phi(t), -h < t < 0. \end{aligned}$$

5. Let $X \sim G(\alpha, \beta)$.

(a) Find mgf of X.

(b) Find $E(X), E(X^2), E(X^3), E(X^4), Var(X)$ using the mgf of X.

(c) Let $X_1, \cdots, X_n$ be independent random variables such that $X_i \sim G(\alpha_i, \beta)$, $i = 1, \cdots, n$. Show that

$$X_1 + \cdots + X_n \sim G(\alpha_1 + \cdots + \alpha_n, \beta).$$