Homework 1

STA 213

Problem 1

Denote F=female, M=male. Then, the sample space Ω is

$$\Omega = \{(M_1, M_2), (F_1, F_2), (M_1, F_2), (F_1, M_2)\}$$

and $\Pr(\omega) = \frac{1}{4}$, for all $\omega \in \Omega$.

Let E be the event that at least one of the children is boy, i.e.

$$E = \{(M_1, M_2), (M_1, F_2), (F_1, M_2)\}.$$

The probability of the event E is

$$\Pr(E) = \Pr[(M_1, M_2)] + \Pr[(F_1, M_2)] + \Pr[(M_1, F_2)] = \frac{3}{4}.$$

The conditional probability that both are boys given that at least one of them is boy is

$$\begin{aligned} \Pr[(M_1, M_2)|E] &= \frac{\Pr[(M_1, M_2)]}{\Pr[E]} \\ &= \frac{1/4}{3/4} = \frac{1}{3}. \end{aligned}$$

Problem 2

Use induction.

1. $n = 2$ trivial

2. Suppose the equality is true for $n-1$. We have to show that it is true also for n.

   Set $E = \cup_{i=1}^{n-1} A_i$. Then

$$
\begin{aligned}
\Pr(\cup_{i=1}^{n} A_i) &= \Pr(E \cup A_n) = \Pr(E) + \Pr(A_n) - \Pr(E \cap A_n) \\
& \quad \text{(because the formular is true for } n = 2) \\
&= \sum_{i=1}^{n-1} \Pr(A_i) + \Pr(A_n) - \sum_{i<j}^{n-1} \Pr(A_i \cap A_j) \\
&+ \sum_{i<j<k}^{n-1} \Pr(A_i \cap A_j \cap A_k) + \cdots + (-1)^n \Pr(\cap_{i=1}^{n-1} A_i) - \Pr(\{\cup_{i=1}^{n-1} A_i\} \cap A_n) \\
& \quad \text{(applied the formular for } n-1 \text{ to } E) \\
&= \sum_{i=1}^{n} \Pr(A_i) - \sum_{i<j}^{n-1} \Pr(A_i \cap A_j) \\
&+ \sum_{i<j<k}^{n-1} \Pr(A_i \cap A_j \cap A_k) + \cdots + (-1)^n \Pr(\cap_{i=1}^{n-1} A_i) - \Pr(\cup_{i=1}^{n-1} \{A_i \cap A_n\}) \\
&= \sum_{i=1}^{n} \Pr(A_i) - \sum_{i<j}^{n-1} \Pr(A_i \cap A_j) \\
&+ \sum_{i<j<k}^{n-1} \Pr(A_i \cap A_j \cap A_k) + \cdots + (-1)^n \Pr(\cap_{i=1}^{n-1} A_i) \\
&- [\sum_{i=1}^{n-1} \Pr(A_i \cap A_n) - \sum_{i<j}^{n-1} \Pr(A_i \cap A_j \cap A_n) + \sum_{i<j<k}^{n-1} \Pr(A_i \cap A_j \cap A_k \cap A_n) \\
&+ \cdots + (-1)^n \Pr(\cap_{i=1}^{n-1} \{A_i \cap A_n\})] \\
& \quad \text{(applied the formular for } n-1 \text{ to } E \cap A_n) \\
&= \sum_{i=1}^{n} \Pr(A_i) - \sum_{i<j}^{n} \Pr(A_i \cap A_j) \\
&+ \sum_{i<j<k}^{n} \Pr(A_i \cap A_j \cap A_k) + \cdots + (-1)^{n+1} \Pr(\cap_{i=1}^{n} A_i). \\
& \quad \text{(by rearranging the terms)}
\end{aligned}
$$

Problem 3

Many other solutions are possible.

Assume that a fair coin is flipped twice and define three events as follows;

$$\begin{aligned} \text{A} &= \text{“head on first flip”} \\ \text{B} &= \text{“head on second flip”} \\ \text{C} &= \text{“same face on both flips”}. \end{aligned}$$

Then

$$\Pr(A) = \Pr(B) = \Pr(C) = \frac{1}{2}$$

and

$$\Pr(A \cap B) = \Pr(A \cap C) = \Pr(B \cap C) = \frac{1}{4};$$

the events A, B, C are pairwise independent. However, they are not mutually independent, because

$$\Pr(A \cap B \cap C) = \frac{1}{4} \neq \left(\frac{1}{2}\right)^3.$$

Problem 10 in p72

(a) Let C_i = “coin i is selected” for $i = 1, \cdots, 5$, H_i = “Head on ith flip” for $i = 1, 2$ and $p_i = \Pr(H_1|C_i)$ for $i = 1, \cdots, 5$.

Note $\Pr(C_i) = \frac{1}{5}$ for $i = 1, 2, 3, 4, 5$.

By Bayes' Theorem,

$$\Pr(C_i|H_1) = \frac{\Pr(H_1|C_i)\Pr(C_i)}{\Pr(H_1)} = \frac{\frac{1}{5}p_i}{\Pr(H_1)},$$

and by Law of Total Probability

$$\Pr(H_1) = \sum_{i=1}^{5} \Pr(H_1|C_i)\Pr(C_i) = \frac{1}{5}\left(0 + \frac{1}{4} + \frac{1}{2} + \frac{3}{4} + 1\right) = \frac{1}{2}.$$

Combining the above two results,

$$\begin{aligned}
\Pr(C_1|H_1) &= 0 \\
\Pr(C_2|H_1) &= 0.1 \\
\Pr(C_3|H_1) &= 0.2 \\
\Pr(C_4|H_1) &= 0.3 \\
\Pr(C_5|H_1) &= 0.4.
\end{aligned}$$

(b)

$$\begin{aligned}
\Pr(H_2|H_1) &= \sum_{i=1}^{5} \Pr(H_2|C_i, H_1) \Pr(C_i|H_1) \\
&= \sum_{i=1}^{5} \Pr(H_2|C_i) \Pr(C_i|H_1) = 0.75 = \frac{3}{4}.
\end{aligned}$$

(c) Let T_1 = "Tail on first flip".
$\Pr(T_1|C_i) = 1 - p_i$ and $\Pr(T_1) = \frac{1}{2}$. Note that

$$\begin{aligned}
\Pr(T_1|C_1) &= p_5 = 1 - p_1 = 1 \\
\Pr(T_1|C_2) &= p_4 = 1 - p_2 = 3/4 \\
\Pr(T_1|C_3) &= p_3 = 1 - p_3 = 1/2 \\
\Pr(T_1|C_4) &= p_2 = 1 - p_4 = 1/4 \\
\Pr(T_1|C_5) &= p_1 = 1 - p_5 = 0.
\end{aligned}$$

Then,

$$\begin{aligned}
\Pr(C_1|T_1) &= \Pr(C_5|H_1) = 0.4 \\
\Pr(C_2|T_1) &= \Pr(C_4|H_1) = 0.3 \\
\Pr(C_3|T_1) &= \Pr(C_3|H_1) = 0.2 \\
\Pr(C_4|T_1) &= \Pr(C_2|H_1) = 0.1 \\
\Pr(C_5|T_1) &= \Pr(C_1|H_1) = 0.
\end{aligned}$$

We obtain

$$\Pr(H_2|T_1) = \sum_{i=1}^{5} \Pr(H_2|C_i) \Pr(C_i|T_1) = \frac{1}{4} = 0.25.$$

Due to the particular structure of this model, we could have obtained the same solution in the following way

$$\begin{aligned}
\Pr(H_2|T_1) &= \sum_{i=1}^{5} \Pr(H_2|C_i)\Pr(C_i|T_1) = \sum_{i=1}^{5} p_i \Pr(C_i|T_1) \\
&= \sum_{i=1}^{5} (1-p_i)\Pr(C_i|H_1) = \sum_{i=1}^{5} \Pr(C_i|H_1) - \sum_{i=1}^{5} p_i \Pr(C_i|H_1) \\
&= 1 - \Pr(H_2|H_1) = 0.25.
\end{aligned}$$

Problem 11 in p72

(a) Define $E = T_1 \cap T_2 \cap T_3 \cap H_4$ and recall $P(C_i) = \frac{1}{5}$, $i = 1, 2, 3, 4, 5.$ Then

$$\Pr(E) = \sum_{i=1}^{5} \Pr(E|C_i)\Pr(C_i) = \frac{1}{5}\sum_{i=1}^{5} p_i(1-p_i)^3 = \frac{23}{640} = 0.0359375.$$

By Bayes' Theorem,

$$\Pr(C_i|E) = \frac{\Pr(E|C_i)\Pr(C_i)}{\Pr(E)} = \frac{\frac{1}{5}p_i(1-p_i)^3}{\Pr(E)}.$$

We obtain

$$\begin{aligned}
\Pr(C_1|E) &= 0 \\
\Pr(C_2|E) &= \frac{27}{46} \approx 0.587 \\
\Pr(C_3|E) &= \frac{8}{23} \approx 0.348 \\
\Pr(C_4|E) &= \frac{6}{92} \approx 0.065 \\
\Pr(C_5|E) &= 0.
\end{aligned}$$

(b) Define $A = T_5 \cap T_6 \cap H_7$.

$$\begin{aligned}
\Pr(A|E) &= \sum_{i=1}^{5} \Pr(A|C_i)\Pr(C_i|E) \\
&= \sum_{i=1}^{5} p_i(1-p_i)^2 \Pr(C_i|E) = \frac{95}{736} \approx 0.129
\end{aligned}$$