Homework 2

STA 213

Problem 1

(a) 1.

$$\begin{array}{ccc} Z = X + Y & & Y = Z - V \\ & \Longrightarrow & \\ V = X & & X = V \end{array}$$

In this case $|J| = 1$. Since $0 < z - v < \infty$, we obtain

$$f(z) = \int_0^z e^{-v} e^{-z+v} dv = ze^{-z}$$

Z has a distribution *Gamma(2,1)*, $z \in (0, +\infty)$.

2.

$$\begin{array}{ccc} Z = X - Y & & X = Z + V \\ & \Longrightarrow & \\ V = Y & & Y = V \end{array}$$

In this case we have that $|J| = 1$, $z \in (-\infty, +\infty)$, $0 < v < +\infty$ and $0 < z+v < +\infty$. Then if $z \geq 0$

$$f(z) = \int_0^{+\infty} e^{-v} e^{-v-z} dv = \frac{e^{-z}}{2}$$

and if $z < 0$

$$f(z) = \int_{-z}^{+\infty} e^{-v} e^{-v-z} dv = \frac{e^{z}}{2}$$

$$\Rightarrow \quad f(z) = \tfrac{1}{2} e^{-|z|} \qquad z \in (-\infty, +\infty)$$

3. $Z = XY$, $z \in (0, \infty)$

$$\begin{aligned}\Pr(Z > z) &= \Pr(XY > z) = E_y\{\Pr(X > z/y | Y = y)\} \\ &= E_y\{\Pr(X > z/y)\} = \int_0^{\infty} e^{-z/y} e^{-y} dy\end{aligned}$$

This integral does not have an analytical solution.

4. $Z = \frac{X}{Y}$, $z \in (0, \infty)$

$$\begin{aligned}\Pr(Z > z) &= \Pr(\frac{X}{Y} > z) = E_y\{\Pr(X > zy | Y = y)\} \\ &= E_y\{\Pr(X > zy)\} = \int_0^{\infty} e^{-zy} e^{-y} dy = \frac{1}{1+z}\end{aligned}$$

Then

$$F(z) = \frac{z}{1+z}$$

and

$$f(z) = \frac{dF(z)}{dz} = \frac{1}{(1+z)^2}, \qquad z \in (0, +\infty)$$

5. $Z = \min\{X, Y\}$

$$f(z) = 2e^{-2z}, \qquad z \in (0, +\infty)$$

Z has an exponential distribution with parameter 2.

6. $Z = \max\{X, Y\}$

$$f(z) = 2(1 - e^{-z})e^{-z}, \qquad z \in (0, +\infty)$$

7. $Y_1 = \min\{X, Y\}$, $Y_2 = \max\{X, Y\}$

$$f(y_1, y_2) = 2e^{-y_1} e^{-y_2} \qquad 0 < y_1 < y_2 < \infty$$

Set

$$\begin{matrix} Z = \frac{Y_1}{Y_2} & & Y_1 = ZV \\ & \Longrightarrow & \\ V = Y_2 & & Y_2 = V \end{matrix}$$

where $z \in (0, 1)$.

We obtain that $|J| = v$. Therefore

$$\begin{aligned}
f(z) &= \int_0^\infty 2ve^{-zv}e^{-v}dv \\
& \quad \text{use integration by parts} \\
&= 2\{[-v\frac{e^{-v(z+1)}}{z+1}]_0^\infty + \int_0^\infty \frac{e^{-v(z+1)}}{z+1}dv\} \\
&= 2\int_0^\infty \frac{e^{-v(z+1)}}{z+1}dv = \frac{2}{(z+1)^2}
\end{aligned}$$

8.

$$\begin{matrix} Z = \frac{X}{X+Y} & & Y = \frac{V(1-Z)}{Z} \\ & \Longrightarrow & \\ V = X & & X = V \end{matrix}$$

where $z \in (0,1)$. In this case $|J| = \frac{v}{z^2}$. Then

$$\begin{aligned}
f(z) &= \int_0^\infty e^{-v}e^{-v(1-z)/z}\frac{v}{z^2}dv = \frac{1}{z^2}\int_0^\infty e^{-v/z}vdv \\
& \quad \text{use integration by parts} \\
&= \frac{1}{z^2}\{[-\frac{ve^{-v/z}}{1/z}]_0^\infty + \int_0^\infty ze^{-v/z}dv\} = 1
\end{aligned}$$

Therefore Z has a *beta distribution* with parameters $\alpha = 1$ and $\beta = 1$, for $0 < z < 1$.

(b)

$$\begin{matrix} U = X + Y & & Y = \frac{U-V}{2} \\ & \Longrightarrow & \\ V = X - Y & & X = \frac{U+V}{2} \end{matrix}$$

In this case $|J| = \frac{1}{2}$. Then

$$f(u,v) = \begin{cases} \frac{1}{2}e^{-\frac{u+v}{2}}e^{-\frac{u-v}{2}} = \frac{1}{2}e^{-u} & \text{for } 0 < u < \infty \text{ and } -u < v < u \\ 0 & \text{otherwise} \end{cases}$$

We obtain

$$f(v|u) = \frac{f(u,v)}{f(v)} = \begin{cases} \frac{1}{2}\frac{e^{-u}}{ue^{-u}} = \frac{1}{2u} & \text{for } -u < v < u \\ 0 & \text{otherwise} \end{cases}$$

(c)

$$U = X + Y \qquad Y = U(1-Z)$$
$$\Longrightarrow$$
$$Z = \frac{X}{X+Y} \qquad X = ZU$$

In this case $|J| = u$. Then

$$f(u,z) = \begin{cases} ue^{-uz}e^{-u(1-z)} = ue^{-u} & \text{for } u > 0 \text{ and } 0 < z < 1 \\ 0 & \text{otherwise} \end{cases}$$

$f(u,z)$ can be appropriately factored and it is positive over an appropriate rectangle; it follows that U and Z are independent.

Problem 2

$$X_1 = (-2\log U_1)^{1/2}\cos(2\pi U_2) \qquad U_1 = \exp\{-\frac{(X_1^2+X_2^2)}{2}\}$$
$$\Longrightarrow$$
$$X_2 = (-2\log U_1)^{1/2}\sin(2\pi U_2) \qquad U_2 = \frac{1}{2\pi}\arctan\frac{X_2}{X_1}$$

The Jacobian of the transformation is:

$$\begin{aligned} |J| &= \begin{vmatrix} -x_1\exp\{-\frac{(x_1^2+x_2^2)}{2}\} & -x_2\exp\{-\frac{(x_1^2+x_2^2)}{2}\} \\ -\frac{1}{2\pi}\frac{1}{1+(\frac{x_2}{x_1})^2}\frac{x_2}{x_1^2} & \frac{1}{2\pi}\frac{1}{1+(\frac{x_2}{x_1})^2}\frac{1}{x_1} \end{vmatrix} \\ &= \exp\{-\frac{(x_1^2+x_2^2)}{2}\}\frac{1}{1+(\frac{x_2}{x_1})^2}\frac{1}{2\pi}(1+(\frac{x_2}{x_1})^2) \\ &= \frac{1}{2\pi}\exp\{-\frac{(x_1^2+x_2^2)}{2}\} \end{aligned}$$

It follows that the joint density of X_1, X_2 is

$$\begin{aligned} f(x_1,x_2) &= \frac{1}{2\pi}\exp\{-\frac{(x_1^2+x_2^2)}{2}\} = \frac{1}{\sqrt{2\pi}}e^{-\frac{x_1^2}{2}}\frac{1}{\sqrt{2\pi}}e^{-\frac{x_1^2}{2}} \\ &= f(x_1)f(x_2), \qquad -\infty < x_1 < \infty, -\infty < x_2 < \infty \end{aligned}$$

Then X_1, X_2 are iid N(0,1).

Problem 3

(a)

$$Z_1 = \frac{X_1-\mu_1}{\sigma_1} \qquad X_1 = \sigma_1 Z_1 + \mu_1$$
$$\Longrightarrow$$
$$Z_2 = \frac{X_2-\mu_2}{\sigma_2} \qquad X_2 = \sigma_2 Z_2 + \mu_2$$

The Jacobian of the transformation is $|J| = \sigma_1\sigma_2$ The joint pdf of (Z_1, Z_2) is

$$\begin{aligned} f(z_1, z_2) &= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\sigma_1\sigma_2 \exp\{-\frac{1}{2}[z_1^2 - 2\rho z_1 z_2 + z_2^2]\} \\ &= \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\{-\frac{1}{2}[z_1^2 - 2\rho z_1 z_2 + z_2^2]\} \\ &\quad \text{for } z_1, z_2 \in R \end{aligned}$$

(b) Marginal of X_1

$$\begin{aligned} f(x_1) &= \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\{-\frac{1}{2(1-\rho^2)}[(\frac{x_1-\mu_1}{\sigma_1})^2 - 2\rho(\frac{x_1-\mu_1}{\sigma_1})(\frac{x_2-\mu_2}{\sigma_2}) + (\frac{x_2-\mu_2}{\sigma_2})^2]\}dx_2 \\ &= \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\{-\frac{1}{2(1-\rho^2)}[(1-\rho^2)(\frac{x_1-\mu_1}{\sigma_1})^2 + \\ &\qquad +\rho^2(\frac{x_1-\mu_1}{\sigma_1})^2 - 2\rho(\frac{x_1-\mu_1}{\sigma_1})(\frac{x_2-\mu_2}{\sigma_2}) + (\frac{x_2-\mu_2}{\sigma_2})^2]\}dx_2 \end{aligned}$$

Set $y = \frac{1}{\sqrt{1-\rho^2}}(\frac{x_2-\mu_2}{\sigma_2} - \rho\frac{x_1-\mu_1}{\sigma_1})$.

$$\Rightarrow x_2 = \sigma_2\sqrt{1-\rho^2}y + \rho\frac{x_1-\mu_1}{\sigma_1}\sigma_2 + \mu_2 \text{ and } \frac{dx_2}{dy} = \sigma_2\sqrt{1-\rho^2}$$

Then

$$\begin{aligned} f(x_1) &= \exp\{-\frac{1}{2}(\frac{x_1-\mu_1}{\sigma_1})^2\} \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\sigma_2\sqrt{1-\rho^2} \exp\{-\frac{1}{2}y^2\}dy \\ &= \frac{1}{\sqrt{2\pi}\sigma_1} \exp\{-\frac{1}{2}(\frac{x_1-\mu_1}{\sigma_1})^2\} \\ &\quad \text{for } x_1 \in R \end{aligned}$$

Same for the marginal of X_2 but use the transformation $z = \frac{1}{\sqrt{1-\rho^2}}(\frac{x_1-\mu_1}{\sigma_1} - \rho\frac{x_2-\mu_2}{\sigma_2})$.

(c)

$$
\begin{aligned}
f(x_1|x_2) &= \frac{f(x_1,x_2)}{f(x_2)} \\
&= \frac{\sqrt{2\pi}\sigma_2}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \frac{\exp\{-\frac{1}{2(1-\rho^2)}[(\frac{x_1-\mu_1}{\sigma_1})^2 - 2\rho(\frac{x_1-\mu_1}{\sigma_1})(\frac{x_2-\mu_2}{\sigma_2}) + (\frac{x_2-\mu_2}{\sigma_2})^2]\}}{\exp\{-\frac{1}{2}(\frac{x_2-\mu_2}{\sigma_2})^2\}} \\
&= \frac{1}{\sqrt{2\pi}\sigma_1\sqrt{1-\rho^2}} \exp\{-\frac{1}{2(1-\rho^2)}[(\frac{x_1-\mu_1}{\sigma_1})^2 - 2\rho(\frac{x_1-\mu_1}{\sigma_1})(\frac{x_2-\mu_2}{\sigma_2}) + \\
&\qquad +(\frac{x_2-\mu_2}{\sigma_2})^2 - (1-\rho^2)(\frac{x_2-\mu_2}{\sigma_2})^2]\} \\
&= \frac{1}{\sqrt{2\pi}\sigma_1\sqrt{1-\rho^2}} \exp\{-\frac{1}{2(1-\rho^2)}[(\frac{x_1-\mu_1}{\sigma_1})^2 - 2\rho(\frac{x_1-\mu_1}{\sigma_1})(\frac{x_2-\mu_2}{\sigma_2}) + \rho^2(\frac{x_2-\mu_2}{\sigma_2})^2]\} \\
&= \frac{1}{\sqrt{2\pi}\sigma_1\sqrt{1-\rho^2}} \exp\{-\frac{1}{2(1-\rho^2)}[\frac{x_1-\mu_1}{\sigma_1} - \rho(\frac{x_2-\mu_2}{\sigma_2})]^2\} \\
&= \frac{1}{\sqrt{2\pi}\sigma_1\sqrt{1-\rho^2}} \exp\{-\frac{1}{2\sigma_1^2(1-\rho^2)}[x_1-\mu_1-\rho\frac{\sigma_1}{\sigma_2}(x_2-\mu_2)]^2\}
\end{aligned}
$$

Therefore $X_1|X_2 = x_2 \sim \mathrm{N}(\mu_1 + \rho\frac{\sigma_1}{\sigma_2}(x_2-\mu_2), \sigma_1^2(1-\rho^2))$.

Problem 4

(a) $Z = X + Y$.

$$
\begin{aligned}
\Pr(Z=z) &= \Pr(X+Y=z) = \sum_{y=0}^{z} \Pr(X=z-y|Y=y)\Pr(Y=y) \\
&= \sum_{y=0}^{z} \Pr(X=z-y)\Pr(Y=y) = \sum_{y=0}^{z} \binom{n}{z-y} p^{z-y}(1-p)^{n-z+y} \binom{n}{y} p^y (1-p)^{n-y} \\
&= p^z(1-p)^{2n-z} \sum_{y=0}^{z} \binom{n}{z-y}\binom{n}{y} \\
&= \binom{2n}{z} p^z (1-p)^{2n-z} \\
&\quad \text{for } z = 0,1,2,\ldots,2n
\end{aligned}
$$

Recall

$$
\sum_{i=0}^{c} \binom{a}{i}\binom{b}{c-i} = \binom{a+b}{c}
$$

(b)

$$\begin{aligned}
\Pr(U=u,V=v) &= \Pr(\frac{X}{Y+1}=u, Y+1=v) = \Pr(X=uv|Y=v-1)\Pr(Y=v-1) \\
&= \Pr(X=uv)\Pr(Y=v-1) = \binom{n}{uv}p^{uv}(1-p)^{n-uv}\binom{n}{v-1}p^{v-1}(1-p)^{n-v+1} \\
&= \binom{n}{uv}\binom{n}{v-1}p^{v(u+1)-1}(1-p)^{2n-v(u+1)+1} \\
&\quad \text{for } v=1,2,\ldots,n+1 \text{ and } u=x/v \text{ where } x=0,1,\ldots,n
\end{aligned}$$

Problem 5

(a)

1. $f_\alpha \geq 0$

   proof:

   It suffices to show that $[1+\alpha(2F_1(x_1)-1)(2F_2(x_2)-1)(2F_3(x_3)-1)] \geq 0$ since $f_1(x_1)f_2(x_2)f_3(x_3) \geq 0$ (the f_i's are pdfs). Note that

   $$-1 \leq 2F_i(x_i) - 1 \leq 1 \qquad\qquad i=1,2,3$$

   Hence

   $$-1 \leq (2F_1(x_1)-1)(2F_2(x_2)-1)(2F_3(x_3)-1) \leq 1$$

   and

   $$1 \geq -\alpha(2F_1(x_1)-1)(2F_2(x_2)-1)(2F_3(x_3)-1)$$

   since $\alpha \in [-1,1]$.

2. $\int f_\alpha = 1$

   proof:

   Note that

   $$\begin{aligned}
   \int f(x_i)F(x_i)dx_i &= \int F'(x_i)F(x_i)dx_i \\
   &\quad \text{use integration by parts} \\
   &= [F(x_i)]^2 - \int f(x_i)F(x_i)dx_i \\
   &\Rightarrow \int f(x_i)F(x_i)dx_i = \frac{[F(x_i)]^2}{2}
   \end{aligned}$$

Hence

$$\iiint\limits_{R\times R\times R} f_1(x_1)f_2(x_2)f_3(x_3)[1+\alpha(2F_1(x_1)-1)(2F_2(x_2)-1)(2F_3(x_3)-1)]dx_1dx_2dx_3 =$$

$$\iiint\limits_{R\times R\times R} f_1(x_1)f_2(x_2)f_3(x_3)dx_1dx_2dx_3 +$$

$$\alpha\iiint\limits_{R\times R\times R} f_1(x_1)f_2(x_2)f_3(x_3)(2F_1(x_1)-1)(2F_2(x_2)-1)(2F_3(x_3)-1)dx_1dx_2dx_3 =$$

$$1+\alpha\int_{-\infty}^{+\infty} f_1(x_1)(2F_1(x_1)-1)dx_1\int_{-\infty}^{+\infty} f_2(x_2)(2F_2(x_2)-1)dx_2\int_{-\infty}^{+\infty} f_3(x_3)(2F_3(x_3)-1)dx_3$$

$$= 1$$

Since

$$\begin{aligned}\int_{-\infty}^{+\infty} f_i(x_i)(2F_i(x_i)-1)dx_i &= \int_{-\infty}^{+\infty} 2f_i(x_i)F_i(x_i)dx_i - \int_{-\infty}^{+\infty} f_i(x_i)dx_i \\ &= [(F_i(x_i))^2]_{-\infty}^{+\infty} - 1 = 0\end{aligned}$$

for $i = 1, 2, 3$.

(b) Marginal of X_1 (same for the marginals of X_2 and X_3).

$$\begin{aligned}f_\alpha(x_1) &= \iint\limits_{R\times R} f_1(x_1)f_2(x_2)f_3(x_3)[1+\alpha(2F_1(x_1)-1)(2F_2(x_2)-1)(2F_3(x_3)-1)]dx_2dx_3 \\ &= f_1(x_1)\iint\limits_{R\times R} f_2(x_2)f_3(x_3)dx_2dx_3 \\ &+ \alpha f_1(x_1)(2F_1(x_1)-1)\iint\limits_{R\times R} f_2(x_2)f_3(x_3)(2F_2(x_2)-1)(2F_3(x_3)-1)]dx_2dx_3 \\ &= f_1(x_1) \\ &+ \alpha f_1(x_1)(2F_1(x_1)-1)\int_{-\infty}^{+\infty} f_2(x_2)(2F_2(x_2)-1)dx_2\int_{-\infty}^{+\infty} f_3(x_3)(2F_3(x_3)-1)dx_3 = f_1(x_1)\end{aligned}$$

Problem 6

Recall some standard results of the theory of order statistics:

- The joint pdf of the order statistics of independent rv's $X_1, \ldots, X_n$ having pdf $f(x)$ is

$$g(x_{(1)}, \ldots, f(x_{(n)}) = n! f(x_{(1)}) \cdots f(x_{(n)}), \qquad -\infty < x_{(1)} < \cdots < x_{(n)} < \infty$$

- Marginal density of $X_{(r)}$

$$g(x_{(r)}) = \frac{n!}{(r-1)!(n-r)!}[F(x_{(r)})]^{r-1}[1-F(x_{(r)})]^{n-r}f(x_{(r)}) \qquad -\infty < x_{(r)} < \infty$$

- The joint pdf of $(X_{(r)}, X_{(s)})$, $s > r$ is

$$g(x_{(r)}, x_{(s)}) = \frac{[F(x_{(r)})]^{r-1}[F(x_{(s)}) - F(x_{(r)})]^{s-r-1}[1-F(x_{(s)})]^{n-s}f(x_{(r)})f(x_{(s)})}{B(r, s-r)B(s, n-s+1)}, \quad -\infty < x_{(r)} < x_{(s)} < \infty$$

where $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ is the *beta function.*

In this case we obtain

$$\begin{aligned} g(x_{(r)}) &= \frac{n!}{(r-1)!(n-r)!}[1-e^{-\beta x_{(r)}}]^{r-1}[e^{-\beta x_{(r)}}]^{n-r}\beta e^{-\beta x_{(r)}} \\ g(x_{(r)}, x_{(s)}) &= \frac{[1-e^{-\beta x_{(r)}}]^{r-1}[e^{-\beta x_{(r)}} - e^{-\beta x_{(s)}}]^{s-r-1}[e^{-\beta x_{(s)}}]^{n-s}\beta^2 e^{-\beta(x_{(r)}+x_{(s)})}}{B(r, s-r)B(s, n-s+1)} \end{aligned}$$

(a)

$$\begin{matrix} Z = X_{(s)} - X_{(r)} & & X_{(s)} = Z + V \\ & \Longrightarrow & \\ V = X_{(r)} & & X_{(r)} = V \end{matrix}$$

In this case $|J| = 1$

$$\begin{aligned} f(v,z) &= \frac{[1-e^{-\beta v}]^{r-1}[e^{-\beta v} - e^{-\beta(v+z)}]^{s-r-1}[e^{-\beta(v+z)}]^{n-s}\beta^2 e^{-\beta(v+v+z)}}{B(r, s-r)B(s, n-s+1)} \\ &= \frac{[1-e^{-\beta v}]^{r-1}[e^{-\beta v}]^{n-r}\beta e^{-\beta v}[1-e^{-\beta z}]^{s-r-1}[e^{-\beta z}]^{n-s}\beta e^{-\beta z}}{B(r, s-r)B(s, n-s+1)} \\ &= \frac{[1-e^{-\beta v}]^{r-1}[e^{-\beta v}]^{n-r}\beta e^{-\beta v}}{B(r, n-r+1)} \frac{[1-e^{-\beta z}]^{s-r-1}[e^{-\beta z}]^{n-s}\beta e^{-\beta z}}{B(s-r, n-s+1)} \end{aligned}$$

for $0 < z < +\infty$ and $0 < v < +\infty$

$f(v,z)$ can be appropriately factored and it is positive over an appropriate rectangle; it follows that V and Z are independent.

(b) $Z = X_{(r+1)} - X_{(r)}$. From (a) we obtain

$$\begin{aligned} f(z) &= \frac{[e^{-\beta z}]^{n-r-1}\beta e^{-\beta z}}{B(1,n-r)} I(z>0) \\ &= \frac{(n-r)!}{(n-r-1)!}[e^{-\beta z}]^{n-r-1}\beta e^{-\beta z} I(z>0) \\ &= (n-r)[e^{-\beta z}]^{n-r-1}\beta e^{-\beta z} I(z>0) \\ &= (n-r)\beta e^{-\beta z} I(z>0) \end{aligned}$$

(c) From (a) we obtain that $X_{(1)}, (X_{(2)} - X_{(1)}), \ldots, (X_{(n)} - X_{(n-1)})$ are independent and

$$\begin{aligned} & f(X_{(1)}, X_{(2)} - X_{(1)}, \ldots, X_{(n)} - X_{(n-1)}) = \\ &= n[e^{-\beta x_{(1)}}]^{n-1}\beta e^{-\beta x_{(1)}}(n-1)[e^{-\beta(x_{(2)}-x_{(1)})}]^{n-2}\beta e^{-\beta(x_{(2)}-x_{(1)})}(n-2)[e^{-\beta(x_{(3)}-x_{(2)})}]^{n-3}\beta e^{-\beta(x_{(3)}-x_{(2)})}\cdots \\ &\quad \cdots \beta e^{-\beta(x_{(n)}-x_{(n-1)})} \\ &= n!\beta^n[e^{-\beta x_{(1)}}]^{n-1}e^{-\beta x_{(1)}}\prod_{j=2}^{n}\{[e^{-\beta(x_{(j)}-x_{(j-1)})}]^{n-j}e^{-\beta(x_{(j)}-x_{(j-1)})}\} \\ &= n!\beta^n[e^{-\beta x_{(1)}}]^{n}\prod_{j=2}^{n}[e^{-\beta(x_{(j)}-x_{(j-1)})}]^{n-j+1} \end{aligned}$$

$$\begin{array}{c} Z_1 = nX_{(1)} \\ Z_2 = (n-1)(X_{(2)} - X_{(1)}) \\ \vdots \\ Z_n = X_{(n)} - X_{(n-1)} \end{array} \quad \Longrightarrow \quad \begin{array}{c} X_{(1)} = \frac{1}{n}Z_1 \\ X_{(2)} - X_{(1)} = \frac{1}{n-1}Z_2 \\ \vdots \\ X_{(n)} - X_{(n-1)} = Z_n \end{array}$$

We obtain

$$|J| = \begin{vmatrix} \frac{1}{n} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{n-1} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{n-2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & 1 \end{vmatrix} = \frac{1}{n!}$$

Then

$$\begin{aligned} f(z_1, z_2, \ldots, z_n) &= \frac{n!}{n!}\beta^n[e^{-\beta\frac{z_1}{n}}]^n\prod_{j=2}^{n}[e^{-\beta\frac{z_j}{n-j+1}}]^{n-j+1} \\ &= \beta^n\prod_{j=1}^{n}\exp\{-\beta z_j\} \end{aligned}$$

Hence $(Z_1, \ldots, Z_n)$ and $(X_1, \ldots, X_n)$ have the same joint distribution.