Homework 2

STA 213

Problem 1
Let

$$X^+ = \max\{X, 0\} \qquad\qquad X^- = -\min\{X, 0\}$$

Then

$$X = X^+ - X^- \qquad\qquad E(X) = E(X^+) - E(X^-)$$

$$\begin{aligned}
E(X) &= \int_0^\infty x dF(x) - \int_{-\infty}^0 (-x) dF(x) \\
&= \int_0^\infty (1 - F(x)) dx - \{-\int_0^\infty x dF(-x)\} \\
&= \int_0^\infty (1 - F(x)) dx - \int_0^\infty F(-x) dx \\
&= \int_0^\infty (1 - F(x)) dx - \int_{-\infty}^0 F(x) dx \\
&= \int_0^\infty \Pr(X > x) dx - \int_{-\infty}^0 \Pr(X \leq x) dx
\end{aligned}$$

$$\begin{aligned}
E|X|^\alpha &= \int_{-\infty}^\infty |x|^\alpha dF(x) \\
&= \int_0^\infty |x|^\alpha dF(x) + \int_{-\infty}^0 |x|^\alpha dF(x)
\end{aligned}$$

Note that

$$\begin{aligned}
\int_0^\infty |x|^\alpha dF(x) &= -\int_0^\infty |x|^\alpha d(1-F(x)) \\
&= \lim_{n\to\infty} -\int_0^n |x|^\alpha d(1-F(x)) \\
&= \lim_{n\to\infty} [-x^\alpha(1-F(x))]_0^n + \lim_{n\to\infty}\int_0^n \alpha x^{\alpha-1}(1-F(x))dx \\
&= \lim_{n\to\infty} -n^\alpha(1-F(n)) + \lim_{n\to\infty}\int_0^n \alpha x^{\alpha-1}(1-F(x))dx \\
&= \alpha\int_0^\infty x^{\alpha-1}(1-F(x))dx
\end{aligned}$$

and

$$\begin{aligned}
\int_{-\infty}^0 |x|^\alpha dF(x) &= -\int_0^\infty x^\alpha dF(-x) \\
&= \lim_{n\to\infty} -\int_0^n x^\alpha dF(-x) \\
&= \lim_{n\to\infty} [-x^\alpha F(-x)]_0^n + \lim_{n\to\infty}\int_0^n \alpha x^{\alpha-1}F(-x)dx \\
&= \lim_{n\to\infty} -n^\alpha F(-n) + \lim_{n\to\infty}\int_0^n \alpha x^{\alpha-1}F(-x)dx \\
&= \alpha\int_0^\infty x^{\alpha-1}F(-x)dx
\end{aligned}$$

Hence

$$\begin{aligned}
E|X|^\alpha &= \int_0^\infty |x|^\alpha dF(x) + \int_{-\infty}^0 |x|^\alpha dF(x) \\
&= \alpha\int_0^\infty x^{\alpha-1}(1-F(x))dx + \alpha\int_0^\infty x^{\alpha-1}F(-x)dx \\
&= \alpha\int_0^\infty x^{\alpha-1}[(1-F(x)) + F(-x)]dx \\
&= \alpha\int_0^\infty x^{\alpha-1}[\Pr(X>x) + \Pr(X\le -x)]dx \\
&= \alpha\int_0^\infty x^{\alpha-1}\Pr(|X|>x)dx
\end{aligned}$$

Problem 2

(a) If $\alpha = 1$,

$$\begin{aligned} x^{\alpha-1}e^{-x} &= e^{-x} \\ \int_0^\infty e^{-x} < +\infty &\Rightarrow \Gamma(\alpha) < +\infty \end{aligned}$$

If $\alpha > 1$ then for x large enough

$$\begin{aligned} x^{\alpha-1}e^{-x} &< e^{-x}e^{x/2} = e^{-x/2} \\ \int_0^\infty e^{-x/2} < +\infty &\Rightarrow \Gamma(\alpha) < +\infty \end{aligned}$$

If $0 < \alpha < 1$

$$\begin{aligned} x^{\alpha-1}e^{-x} &< e^{-x} \\ \int_0^\infty e^{-x} < +\infty &\Rightarrow \Gamma(\alpha) < +\infty \end{aligned}$$

(b)

$$\begin{aligned} \Gamma(\alpha) &= \int_0^\infty x^{\alpha-1}e^{-x}dx = [-x^{\alpha-1}e^{-x}]_0^\infty + \int_0^\infty (\alpha-1)x^{\alpha-2}e^{-x}dx \\ &= (\alpha-1)\Gamma(\alpha-1) \end{aligned}$$

(c) If α is a positive integer n, we may apply the recurrence formula in (b) to obtain

$$\Gamma(n) = (n-1)(n-2)\cdots(2)(1)\Gamma(1)$$

and since

$$\Gamma(1) = \int_0^\infty e^{-x}dx = 1$$

we have

$$\Gamma(n) = (n-1)!$$

(d) Set $y = \beta x \Rightarrow x = \frac{y}{\beta}$ and $dx = \frac{dy}{\beta}$. We obtain

$$\begin{aligned} \int_0^\infty x^{\alpha-1}e^{-\beta x}dx &= \int_0^\infty \left(\frac{y}{\beta}\right)^{\alpha-1} e^{-y}\frac{dy}{\beta} \\ &= \frac{\Gamma(\alpha)}{\beta^\alpha} \end{aligned}$$

(e) Define

$$I = \int_{-\infty}^{+\infty} e^{-\frac{1}{2}x^2} dx$$

To show that $I = \sqrt{(2\pi)}$ it suffices to show that $I^2 = 2\pi$.

$$\begin{aligned} I^2 &= \int_{-\infty}^{+\infty} e^{-\frac{1}{2}x^2} dx \int_{-\infty}^{+\infty} e^{-\frac{1}{2}y^2} dy \\ &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}(x^2+y^2)} dx dy \end{aligned}$$

Use the polar coordinate transformation:

$$\begin{aligned} x &= r \sin \theta \\ y &= r \cos \theta \end{aligned}$$

and the integral becomes

$$\begin{aligned} I^2 &= \int_{0}^{+\infty} \int_{0}^{2\pi} r e^{-\frac{1}{2}r^2} d\theta dr \\ &= \int_{0}^{+\infty} 2\pi r e^{-\frac{1}{2}r^2} dr \\ &= 2\pi \qquad\qquad \Rightarrow \\ I &= \sqrt{2\pi} \end{aligned}$$

(f)

$$\Gamma(\frac{1}{2}) = \int_{0}^{\infty} x^{-\frac{1}{2}} e^{-x} dx$$

Let $x = y^2/2$. Then $dx = ydy$ and

$$\begin{aligned} \Gamma(\frac{1}{2}) &= \int_{0}^{\infty} x^{-\frac{1}{2}} e^{-x} dx \\ &= \int_{0}^{\infty} \left(\frac{y^2}{2}\right)^{-\frac{1}{2}} e^{-\frac{1}{2}y^2} y dy \\ &= \int_{0}^{\infty} \sqrt{2} e^{-\frac{1}{2}y^2} dy \\ &= \sqrt{2}\frac{\sqrt{2\pi}}{2} = \sqrt{\pi} \end{aligned}$$

(g)

$$
\begin{aligned}
\Gamma(\alpha)\Gamma(\beta) &= \int_0^\infty x^{\alpha-1}e^{-x}dx \int_0^\infty y^{\beta-1}e^{-y}dy \\
&= \int_0^\infty \int_0^\infty x^{\alpha-1}y^{\beta-1}e^{-(x+y)}dxdy
\end{aligned}
$$

Let

$$
u = \frac{x}{x+y} \qquad \Rightarrow x = \frac{uy}{1-u} \qquad \text{and } dx = \frac{ydu}{(1-u)^2}
$$

Then

$$
\begin{aligned}
\Gamma(\alpha)\Gamma(\beta) &= \int_0^\infty \int_0^1 \left(\frac{uy}{1-u}\right)^{\alpha-1} y^{\beta-1}e^{-\frac{y}{1-u}}\frac{y}{(1-u)^2}dudy \\
&= \int_0^\infty \int_0^1 \frac{u^{\alpha-1}}{(1-u)^{\alpha+1}}y^{\alpha+\beta-1}e^{-\frac{y}{1-u}}dudy
\end{aligned}
$$

Let

$$
y = (1-u)v \qquad \Rightarrow dy = (1-u)dv
$$

$$
\begin{aligned}
\Gamma(\alpha)\Gamma(\beta) &= \int_0^\infty \int_0^1 \frac{u^{\alpha-1}}{(1-u)^{\alpha+1}}(1-u)^{\alpha+\beta-1}v^{\alpha+\beta-1}e^{-v}(1-u)dudv \\
&= \int_0^\infty \int_0^1 u^{\alpha-1}(1-u)^{\beta-1}v^{\alpha+\beta-1}e^{-v}dudv \\
&= \int_0^1 u^{\alpha-1}(1-u)^{\beta-1}du \int_0^\infty v^{\alpha+\beta-1}e^{-v}dv \\
&= \Gamma(\alpha+\beta)\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}du \\
&\Downarrow \\
\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} &= \int_0^1 u^{\alpha-1}(1-u)^{\beta-1}du
\end{aligned}
$$

(h)

$$
\prod_{i=1}^k \Gamma(\alpha_i) = \int_0^\infty \cdots \int_0^\infty \prod_{i=1}^k x_i^{\alpha_i-1}e^{-\sum_{i=1}^k x_i}dx_1\cdots dx_{k-1}dx_k
$$

Let

$$
\begin{array}{lcll}
y = \sum_{i=1}^k x_i & & x_i = yy_i & i = 1,\ldots,k-1 \\
 & \Longrightarrow & & \\
y_i = \frac{x_i}{\sum_{i=1}^k x_i} \quad i = 1,\ldots,k-1 & & x_k = y(1-\sum_{i=1}^{k-1} y_i) &
\end{array}
$$

The Jacobian of the transformation is

$$
\begin{aligned}
|J| &= \frac{\partial(x_1, x_2, \ldots, x_k)}{\partial(y_1, \ldots, y_{k-1}, y)} \\
&= \begin{vmatrix}
y & 0 & 0 & \cdots & 0 & y_1 \\
0 & y & 0 & \cdots & 0 & y_2 \\
0 & 0 & y & \cdots & 0 & y_3 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & y & y_{k-1} \\
-y & -y & -y & \cdots & -y & 1 - \sum_{i=1}^{k-1} y_i
\end{vmatrix} \\
&= y^{k-1}
\end{aligned}
$$

We obtain

$$
\begin{aligned}
\prod_{i=1}^{k} \Gamma(\alpha_i) &= \underset{\{y>0; y_i>0,\ i=1,\ldots,k-1; \sum_{i=1}^{k-1} y_i<1\}}{\int \cdots \int} \prod_{i=1}^{k-1} (y y_i)^{\alpha_i - 1} [y(1 - \sum_{i=1}^{k-1} y_i)]^{\alpha_k - 1} e^{-y} y^{k-1} dx_1 \cdots dx_{k-1} dx_k \\
&= \underset{\{y>0; y_i>0,\ i=1,\ldots,k-1; \sum_{i=1}^{k-1} y_i<1\}}{\int \cdots \int} \left(\prod_{i=1}^{k-1} y_i^{\alpha_i - 1} \right) (1 - \sum_{i=1}^{k-1} y_i)^{\alpha_k - 1} y^{\sum_{i=1}^{k} \alpha_i - 1} e^{-y} dx_1 \cdots dx_{k-1} dx_k \\
&= \underset{\{y_i>0,\ i=1,\ldots,k-1; \sum_{i=1}^{k-1} y_i<1\}}{\int \cdots \int} \prod_{i=1}^{k-1} y_i^{\alpha_i - 1} (1 - \sum_{i=1}^{k-1} y_i)^{\alpha_k - 1} dx_1 \cdots dx_{k-1} \int_0^\infty y^{\sum_{i=1}^{k} \alpha_i - 1} e^{-y} dx_k \\
&= \underset{\{y_i>0,\ i=1,\ldots,k-1; \sum_{i=1}^{k-1} y_i<1\}}{\int \cdots \int} \prod_{i=1}^{k-1} y_i^{\alpha_i - 1} (1 - \sum_{i=1}^{k-1} y_i)^{\alpha_k - 1} dx_1 \cdots dx_{k-1} \Gamma\left(\sum_{i=1}^{k} \alpha_i \right) \\
&\Downarrow \\
\frac{\prod_{i=1}^{k} \Gamma(\alpha_i)}{\Gamma(\sum_{i=1}^{k} \alpha_i)} &= \underset{\{y_i>0,\ i=1,\ldots,k-1; \sum_{i=1}^{k-1} y_i<1\}}{\int \cdots \int} \prod_{i=1}^{k-1} y_i^{\alpha_i - 1} (1 - \sum_{i=1}^{k-1} y_i)^{\alpha_k - 1} dx_1 \cdots dx_{k-1}
\end{aligned}
$$

Problem 3

Let

$$
Y = |X|^r \qquad Z = 1 \qquad p = \frac{s}{r} \qquad q = \frac{s}{s-r}
$$

Then $p, q > 1$ and $(1/p) + (1/q) = 1$. By Holder inequality

$$\begin{aligned}
E(|YZ|) &\leq (E(|Y|^p)^{1/p}(E(|Z|^q)^{1/q} \\
&\Downarrow \\
E(|X|^r) &\leq (E(|X|^{r\frac{s}{r}}))^{\frac{r}{s}} \\
&\Downarrow \\
(E(|X|^r))^{1/r} &\leq (E(|X|^s))^{1/s}
\end{aligned}$$

Problem 4

If $0 < t < h$, then

$$\begin{aligned}
\phi(t) &= \int_{-\infty}^{\infty} e^{tx} dF(x) \\
&= \int_{-\infty}^{a} e^{tx} dF(x) + \int_{a}^{\infty} e^{tx} dF(x) \\
&\geq \int_{a}^{\infty} e^{tx} dF(x) \\
&\geq e^{ta} \int_{a}^{\infty} dF(x) = e^{ta} \Pr(X \geq a) \\
&\Downarrow \\
\Pr(X \geq a) &\leq e^{-ta} \phi(t)
\end{aligned}$$

If $-h < t < 0$,then

$$\begin{aligned}
\phi(t) &= \int_{-\infty}^{\infty} e^{tx} dF(x) \\
&= \int_{-\infty}^{a} e^{tx} dF(x) + \int_{a}^{\infty} e^{tx} dF(x) \\
&\geq \int_{-\infty}^{a} e^{tx} dF(x) \\
&\geq e^{ta} \int_{-\infty}^{a} dF(x) = e^{ta} \Pr(X \leq a) \\
&\Downarrow \\
\Pr(X \leq a) &\leq e^{-ta} \phi(t)
\end{aligned}$$

Problem 5

(a)

$$
\begin{aligned}
\psi(t) &= \int_0^{+\infty} e^{tx}\frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}dx \\
&= \left(\frac{\beta}{\beta-t}\right)^\alpha \qquad\qquad \text{for } t<\beta
\end{aligned}
$$

(b)

$$
\begin{aligned}
E(X) &= \psi'(0) = \left.\frac{\alpha\beta^\alpha}{(\beta-t)^{\alpha+1}}\right|_{t=0} = \frac{\alpha}{\beta} \\
E(X^2) &= \psi''(0) = \left.\frac{\alpha(\alpha+1)\beta^\alpha}{(\beta-t)^{\alpha+2}}\right|_{t=0} = \frac{\alpha(\alpha+1)}{\beta^2} \\
E(X^3) &= \psi^{(3)}(0) = \left.\frac{\alpha(\alpha+1)(\alpha+2)\beta^\alpha}{(\beta-t)^{\alpha+3}}\right|_{t=0} = \frac{\alpha(\alpha+1)(\alpha+2)}{\beta^3} \\
E(X^4) &= \psi^{(4)}(0) = \left.\frac{\alpha(\alpha+1)(\alpha+2)(\alpha+3)\beta^\alpha}{(\beta-t)^{\alpha+4}}\right|_{t=0} = \frac{\alpha(\alpha+1)(\alpha+2)(\alpha+3)}{\beta^4}
\end{aligned}
$$

(c) Let $\psi_i(t)$ denote the m.g.f. of X_i, $i=1,2,\ldots,n$ and let $\psi(t)$ be the m.g.f. of the sum $X_1+X_2+\cdots+X_n$. Then

$$
\psi(t) = \prod_i^n \psi_i(t) = \prod_i^n \left(\frac{\beta}{\beta-t}\right)^{\alpha_i} = \left(\frac{\beta}{\beta-t}\right)^{\alpha_1+\cdots+\alpha_n}
$$

$\sum_i^n X_i$ has a distribution $Gamma(\alpha_1+\cdots+\alpha_n,\beta)$.