Homework 4

STA 213

Problem 1

(a)

$$\begin{array}{ccc} Z = X + Y & & X = \frac{VZ}{1+V} \\ & \Longrightarrow & \\ V = \frac{X}{Y} & & Y = \frac{Z}{1+V} \end{array}$$

The Jacobian is

$$|J| = \begin{vmatrix} \frac{v}{1+v} & \frac{z}{(1+v)^2} \\ \frac{1}{1+v} & -\frac{z}{(1+v)^2} \end{vmatrix} = \frac{z}{(1+v)^2}$$

$$\begin{aligned} f(z,v) &= \frac{\beta^{\alpha_1}}{\Gamma(\alpha_1)} \left(\frac{vz}{(1+v)} \right)^{\alpha_1 - 1} \exp\{-\beta \frac{vz}{(1+v)}\} \frac{\beta^{\alpha_2}}{\Gamma(\alpha_2)} \left(\frac{z}{(1+v)} \right)^{\alpha_2 - 1} \\ & \quad \cdot \exp\{-\beta \frac{z}{(1+v)}\} \frac{z}{(1+v)^2} I(0 < z < \infty) I(0 < v < \infty) \\ &= \frac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} z^{\alpha_1+\alpha_2-1} e^{-\beta z} I(0 < z < \infty) I(0 < v < \infty) v^{\alpha_1 - 1} (1+v)^{-\alpha_1-\alpha_2} \end{aligned}$$

$f(z,v)$ can be appropriately factored and it is positive over an appropriate rectangle; it follows that Z and V are independent.

(b)

$$\begin{array}{ccc} U = X + Y & & Y = U(1-Z) \\ & \Longrightarrow & \\ Z = \frac{X}{X+Y} & & X = ZU \end{array}$$

In this case $|J| = u$. Then

$$\begin{aligned} f(u,z) &= \frac{\beta^{\alpha_1}}{\Gamma(\alpha_1)} (zu)^{\alpha_1 - 1} e^{-\beta z u} \frac{\beta^{\alpha_2}}{\Gamma(\alpha_2)} [u(1-z)]^{\alpha_2 - 1} e^{-\beta u(1-z)} u I(u > 0) I(0 < z < 1) \\ &= \frac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} u^{\alpha_1+\alpha_2-1} e^{-\beta u} z^{\alpha_1-1} (1-z)^{\alpha_2-1} I(u > 0) I(0 < z < 1) \\ &= \frac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2)} u^{\alpha_1+\alpha_2-1} e^{-\beta u} I(u > 0) \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} z^{\alpha_1-1} (1-z)^{\alpha_2-1} I(0 < z < 1) \end{aligned}$$

$f(u, z)$ can be appropriately factored and it is positive over an appropriate rectangle; it follows that U and Z are independent.

(c) From point (b) $X + Y \sim Gamma(\alpha_1 + \alpha_2, \beta)$ and $X/(X + Y) \sim Beta(\alpha_1, \alpha_2)$.

Problem 2

(a) Consider two independent random variables Y and Z, such that Y has a standard normal distribution and Z has χ^2 distribution with n degrees of freedom. Then

$$X = \frac{Y}{\sqrt{\frac{Z}{n}}}$$

has a t distribution with n degrees of freedom.
Derivation.

$$f(y, z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} \frac{1}{2^{n/2}\Gamma\left(\frac{n}{2}\right)} z^{(n/2)-1} e^{-z/2}$$

Set

$$\begin{array}{ccc} X = \frac{Y}{\sqrt{\frac{Z}{n}}} & & Y = \frac{1}{n^{1/2}} X W^{1/2} \\ & \Longrightarrow & \\ W = Z & & Z = W \end{array}$$

In this case $|J| = (W/n)^{1/2}$. We obtain

$$\begin{aligned} f(x, w) &= \frac{1}{\sqrt{2\pi}} \frac{1}{2^{n/2}\Gamma\left(\frac{n}{2}\right)} \exp\{-\frac{1}{2n} x^2 w\} w^{(n/2)-1} \exp\{-w/2\} \left(\frac{w}{n}\right)^{1/2} \\ &= \frac{1}{\sqrt{n\pi}} \frac{1}{2^{(n+1)/2}\Gamma\left(\frac{n}{2}\right)} w^{(n-1)/2} \exp\{-\frac{1}{2} w [\frac{1}{n} x^2 + 1]\} \end{aligned}$$

The marginal of X is given by

$$\begin{aligned} f(x) &= \int_0^\infty f(x, z) dz = \\ &= \frac{1}{\sqrt{n\pi}} \frac{1}{2^{(n+1)/2}\Gamma\left(\frac{n}{2}\right)} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\left[\frac{1}{2}\left(1 + \frac{x^2}{n}\right)\right]^{(n+1)/2}} \\ &= \frac{1}{\sqrt{n\pi}} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \left(1 + \frac{x^2}{n}\right)^{-(n+1)/2} \end{aligned}$$

for $-\infty < x < \infty$.

(b) The t distribution is a unimodal distribution symmetrical about $x = 0$. All finite moments of odd order about the origin are therefore zero. Thus $E(X) = 0$. Note that $E(X)$ does not exist for $n = 1$.

$$V(X) = E(X^2)$$

$$\begin{aligned} E(X^2) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{n\pi}} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} x^2 \left(1 + \frac{x^2}{n}\right)^{-(n+1)/2} dx \\ &= 2\int_{0}^{\infty} \frac{1}{\sqrt{n\pi}} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} x^2 \left(1 + \frac{x^2}{n}\right)^{-(n+1)/2} dx \end{aligned}$$

Set

$$y = \frac{\frac{x^2}{n}}{1 + \frac{x^2}{n}} \quad \Longrightarrow \quad x = \sqrt{\frac{yn}{1-y}}$$

Then

$$\frac{dx}{dy} = \frac{1}{2}\left(\frac{yn}{1-y}\right)^{-\frac{1}{2}} \frac{n}{(1-y)^2}$$

$$\begin{aligned} E(X^2) &= 2\int_{0}^{1} \frac{1}{\sqrt{n\pi}} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \frac{yn}{1-y} \left(1 + \frac{1}{n}\frac{yn}{1-y}\right)^{-(n+1)/2} \frac{1}{2}\left(\frac{yn}{1-y}\right)^{-\frac{1}{2}} \frac{n}{(1-y)^2} dy \\ &= \int_{0}^{1} \frac{1}{\sqrt{n\pi}} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \left(\frac{yn}{1-y}\right)^{1-\frac{1}{2}} \frac{n}{(1-y)^2} \left(\frac{1}{1-y}\right)^{-\frac{n+1}{2}} dy \\ &= n\sqrt{n}\frac{1}{\sqrt{n\pi}} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \int_{0}^{1} y^{1/2}(1-y)^{\frac{n-4}{2}} dy \\ &= n\sqrt{n}\frac{1}{\sqrt{n\pi}} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{n}{2}-1\right)}{\Gamma\left(\frac{n+1}{2}\right)} \\ &= \frac{1}{2}\frac{n\sqrt{\pi}\,\Gamma\left(\frac{n}{2}-1\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n}{2}\right)} \\ &= \frac{n}{2}\frac{1}{\left(\frac{n}{2}-1\right)} \\ &= \frac{n}{n-2} \end{aligned}$$

Note that for $n \leq 2$ the integral does not converge.

Problem 3

(a) Consider two independent random variables Y and Z such that Y has a χ^2 distribution with m degrees of freedom and Z has a χ^2 distribution with n degrees of freedom, where m and n are given positive integers. Then

$$X = \frac{Y/m}{Z/n} = \frac{nY}{mZ}$$

has an F distribution with m and n degrees of freedom.
Derivation.

$$f(y,z) = \frac{1}{2^{(m+n)/2}\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} y^{(m/2)-1} z^{(n/2)-1} e^{-(y+z)/2}$$

Set

$$x = \frac{ny}{mz} \quad \Longrightarrow \quad y = \frac{mxz}{n}$$

Then

$$\frac{dy}{dx} = \frac{mz}{n}$$

We obtain

$$f(x,z) = \frac{1}{2^{(m+n)/2}\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} \left(\frac{m}{n}\right)^{m/2} x^{(m/2)-1} z^{\frac{m+n}{2}-1} \exp\{-\frac{1}{2}\left(\frac{m}{n}x+1\right)z\}$$

The marginal of X is given by

$$\begin{aligned}
f(x) &= \int_0^\infty f(x,z)dz \\
&= \frac{1}{2^{(m+n)/2}\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} \left(\frac{m}{n}\right)^{m/2} x^{(m/2)-1} \int_0^\infty z^{\frac{m+n}{2}-1} \exp\{-\frac{1}{2}\left(\frac{m}{n}x+1\right)z\}dz \\
&= \frac{1}{2^{(m+n)/2}\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} \left(\frac{m}{n}\right)^{m/2} x^{(m/2)-1} \frac{\Gamma\left(\frac{m+n}{2}\right)}{\left(\frac{1}{2}\left[\frac{m}{n}x+1\right]\right)^{\frac{m+n}{2}}} \\
&= \frac{\Gamma\left(\frac{m+n}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} \left(\frac{m}{n}\right)^{m/2} \frac{x^{(m/2)-1}}{(mx+n)^{\frac{m+n}{2}}} n^{\frac{m+n}{2}} \\
&= \frac{\Gamma\left(\frac{m+n}{2}\right) m^{m/2} n^{n/2}}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} \frac{x^{(m/2)-1}}{(mx+n)^{\frac{m+n}{2}}}
\end{aligned}$$

for $x > 0$.

(b)

$$E(X) = \int_0^\infty \frac{\Gamma\left(\frac{m+n}{2}\right) m^{m/2} n^{n/2}}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} x \frac{x^{(m/2)-1}}{(mx+n)^{\frac{m+n}{2}}} dx$$

Set

$$y = \frac{\frac{m}{n}x}{1+\frac{m}{n}x} \qquad \Longrightarrow \qquad x = \frac{n}{m}\frac{y}{1-y}$$

Then

$$\frac{dx}{dy} = \frac{n}{m}\frac{1}{(1-y)^2}$$

We obtain

$$\begin{aligned}
E(X) &= \int_0^\infty \frac{\Gamma\left(\frac{m+n}{2}\right) m^{m/2} n^{n/2}}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} \frac{x^{m/2}}{(mx+n)^{\frac{m+n}{2}}} dx \\
&= \int_0^1 \frac{\Gamma\left(\frac{m+n}{2}\right) m^{m/2} n^{n/2}}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} \left(\frac{m}{n}\right)^{\frac{m}{2}} \left(\frac{y}{1-y}\right)^{\frac{m}{2}} \frac{1}{\left(m\frac{n}{m}\frac{y}{1-y}+n\right)^{\frac{m+n}{2}}} \frac{1}{(1-y)^2}\frac{n}{m} dy \\
&= \int_0^1 \frac{\Gamma\left(\frac{m+n}{2}\right) m^{m/2} n^{n/2}}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} m^{-\frac{m}{2}-1} n^{-\frac{n}{2}+1} y^{m/2}(1-y)^{\frac{n}{2}-2} dy \\
&= \frac{\Gamma\left(\frac{m+n}{2}\right) m^{-1} n}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} \frac{\Gamma\left(\frac{m}{2}+1\right)\Gamma\left(\frac{n}{2}-1\right)}{\Gamma\left(\frac{m+n}{2}\right)} \\
&= \frac{n}{m}\frac{\frac{m}{2}}{\left(\frac{n}{2}-1\right)} \\
&= \frac{n}{n-2}
\end{aligned}$$

Note that the integral does not converge for $n \leq 2$.

$$E(X^2) = \int_0^\infty \frac{\Gamma\left(\frac{m+n}{2}\right) m^{m/2} n^{n/2}}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} x^2 \frac{x^{(m/2)-1}}{(mx+n)^{\frac{m+n}{2}}} dx$$

Use the same change of variable as before. We obtain

$$\begin{aligned}
E(X^2) &= \int_0^1 \frac{\Gamma\left(\frac{m+n}{2}\right) m^{m/2} n^{n/2}}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} \left(\frac{n}{m}\right)^{\frac{m}{2}+1} y^{\frac{m}{2}+1} \left(\frac{1}{1-y}\right)^{\frac{m}{2}+1} \frac{1}{\left(m\frac{n}{m}\frac{y}{1-y}+n\right)^{\frac{m+n}{2}}} \frac{1}{(1-y)^2}\frac{n}{m}dy \\
&= \int_0^1 \frac{\Gamma\left(\frac{m+n}{2}\right) m^{m/2} n^{n/2}}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} m^{-\frac{m}{2}-2} n^{-\frac{n}{2}+2} y^{\frac{m}{2}+1}(1-y)^{\frac{n}{2}-3}dy \\
&= \frac{\Gamma\left(\frac{m+n}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} m^{-2} n^2 \frac{\Gamma\left(\frac{m}{2}+2\right)\Gamma\left(\frac{n}{2}-2\right)}{\Gamma\left(\frac{m+n}{2}\right)} \\
&= \frac{m^{-2}n^2}{\left(\frac{n}{2}-1\right)\left(\frac{n}{2}-2\right)}\left(\frac{m}{2}+1\right)\frac{m}{2} \\
&= \frac{n^2(m+2)}{m(n-2)(n-4)}
\end{aligned}$$

Note that the integral does not converge for $n \leq 4$.

$$\begin{aligned}
V(X) &= \frac{n^2(m+2)}{m(n-2)(n-4)} - \frac{n^2}{(n-2)^2} \\
&= \frac{n^2(2m+2n-4)}{m(n-2)^2(n-4)}
\end{aligned}$$

(c)

$$Y = \frac{1}{1+\frac{m}{n}X} \Longrightarrow X = \frac{n}{m}\frac{1-Y}{Y}$$

In this case

$$\left|\frac{dx}{dy}\right| = \frac{n}{mY^2}$$

We obtain

$$\begin{aligned}
f(y) &= \frac{\Gamma(\frac{1}{2}(m+n))}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})} m^{m/2} n^{n/2} \left(\frac{n}{m}\right)^{\frac{m}{2}-1} \left(\frac{1-y}{y}\right)^{\frac{m}{2}-1} \frac{1}{\left(\frac{n(1-y)}{y}+n\right)^{\frac{m+n}{2}}} \frac{n}{mY^2} I(0<y<1) \\
&= \frac{\Gamma(\frac{1}{2}(m+n))}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})} n^{\frac{n+m}{2}} (1-y)^{\frac{m}{2}-1} \left(\frac{1}{y}\right)^{\frac{m}{2}+1} \frac{1}{n^{\frac{n+m}{2}}} y^{\frac{n+m}{2}} I(0<y<1) \\
&= \frac{\Gamma(\frac{1}{2}(m+n))}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})} (1-y)^{\frac{m}{2}-1} y^{\frac{n}{2}-1} I(0<y<1)
\end{aligned}$$

Thus $Y \sim Beta\left(\frac{n}{2}, \frac{m}{2}\right)$.
Alternatively you can simply notice that

$$\frac{1}{1+\frac{m}{n}X} = \frac{Z}{Z+Y}$$

and from (1.c) you can deduce that $\frac{Z}{Z+Y} \sim Beta\left(\frac{n}{2}, \frac{m}{2}\right)$.